What is the infinite?

ØYSTEIN LINNEBO CHECKS IN TO HILBERT'S HOTEL

The famous mathematician David Hilbert introduces our topic as follows in his essay "On the Infinite": "The infinite has always stirred the emotions of mankind more deeply than any other questions; the infinite has stimulated and fertilised reason as few other ideas have; but also the infinite, more than any other notion, is in need of clarification."

Before trying to answer whether there is such a thing as the infinite, let's do as Hilbert recommends and clarify the concept. I shall do so by charting its development, paying special attention to two conceptual revolutions. Throughout, I hope to show how thinking about the infinite is not only interesting but also fun, not least because of the way considerations about mathematics and philosophy (as well as theology, for those so inclined) are woven together.

To put the infinite in perspective, let's begin with some examples of very large finite quantities. Space provides some good examples. A trip around the equator is about 40,000 km. The distance to the moon is about ten times as large. The distance to Mars is about 143 times larger still. Now we take a big jump. The nearest star other than the sun, Alpha Centauri, is about 4.2 lightyears away and thus 750,000 times as remote as Mars. The nearest other galaxy is about 6,000 times farther still, namely 25,000 lightyears. And of course, all these huge distances are nothing compared with the universe itself, which extends far beyond what I have described.

So are space and time infinite in extent? It may seem obvious that they are. No matter how far you travel in any chosen direction, you will never reach a boundary or an "end of space". You will always be able to go on. But this doesn't actually prove that space is infinite. To see this, consider an ordinary two-dimensional sphere, such as the surface of the earth. Here too one can travel indefinitely far in any given direction without ever encountering a boundary. The surface of the sphere is nevertheless finite in area. An analogous possibility exists for the universe as a whole. It's mathematically possible that the universe is geometrically more like a sphere than a plane and thus, although without boundaries, finite in extent. Whether or not this mathematical possibility describes our real universe is an empirical question for cosmology, not for mathematics or philosophy. It turns out that different cosmological models yield different answers. The universe appears to be finely balanced between being such as to make space and time infinite and being such as to make both finite.

In order to find clearer and less controversial examples of infinities, let's leave the messy physical world and turn to the serene and abstract world of pure mathematics. Consider for instance the natural numbers, that is, the numbers 1, 2, 3, etc. For every natural number, we can describe a larger number simply by adding one. We can get to large numbers more quickly by describing larger jumps up the number sequence. Multiplication provides an easy way to describe fairly large numbers. Exponentiation provides significantly larger jumps. Consider for example 10^{80}, that is, the result of multiplying 10 with itself 80 times. This is a remarkably large number: It is larger than the number of atoms in the (observable) universe. But with some cleverness we can do even better.

There is an operation, known as *super-exponentiation*, which stands to exponentiation as this stands to multiplication. For instance, 10 to the super-exponent 3 – written superexp(10,3) – is

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10 to the power of 10^{10}. And superexp(10,80) is a truly gigantic number – so large that, if written out in ordinary decimal notation, it would require more zeros than there are atoms in the universe! Yet even this gigantic number is vanishingly small when compared with the totality of the natural numbers. Imagine there is an infinite queue and you're in position superexp(10,80). Although you no doubt face a long wait, your position is actually uncharacteristically good. Because there are infinitely many natural numbers, the proportion of people in the queue whose position is better than yours is superexp(10,80) divided by infinity, which is zero!

The word "infinite" is derived from "without any limit". This provides an apt characterisation of the ancient concept of infinity as well. To be infinite (*apeiron*) was to be without a limit or bound. By contrast, the contemporary concept of infinity is that of being larger than any natural number. For instance, to say that there are infinitely many stars is to say that for any natural number n, there are more than n stars.

How are these two concepts of infinity – unboundedness and being larger than any natural number – related? For millennia, the two concepts were thought to be coextensive. One implication is fairly straightforward. If a magnitude is unbounded, then its size cannot be measured by any natural number, as this number would otherwise provide a bound. But the reverse implication is problematic. Assume that some magnitude is not bounded by any natural number. Does it follow that the magnitude is

It's possible that the universe is more like a sphere than a plane

*un*bounded? It is useful to think of the bounds as measuring sticks that are longer than the object to be measured. Our question is then whether there can be measuring sticks longer than those provided by the natural numbers. For millennia the answer was assumed to be *no*: everything that is bounded at all is bounded by a natural number. We shall see that this is false. The last 130 years have seen the discovery of some enormous measuring sticks, much longer than those provided by any natural number.

The ancients drew an important distinction between actual and potential infinity. An infinity is *actual* if all of it actually exists. For instance, an infinity of currently existing atoms would be an actual infinity. By contrast, an infinity is *merely potential* if it is tied to some unlimited possibility of going on. A nice example of Aristotle's is the infinite divisibility of matter. Consider a stick. No matter how many times you have bisected the stick, it's possible to do so again – perhaps not *in practice*, as your knife may be too dull, but it's certainly possible *in principle*.

Aristotle held that there can be no actual infinities, only potential ones. This view was an ingenious attempt to reconcile two apparently conflicting views. On the one hand, Aristotle thought that actual infinities lead to paradox, in part for reasons related to Zeno's famous paradoxes. (Some later arguments to the same effect are considered below.) On the other hand, there appear to be examples of the infinite, for instance space, time, and the natural numbers. The paradoxes are avoided by denying the actual infinite, while the apparent examples of the infinite are analysed as involving only the potentially infinite.

The transition from the ancient concept of infinity to our contemporary one saw two major conceptual revolutions: one sensible, another problematic.

As mentioned a moment ago, actual infinities were thought to lead to paradoxes. The famous thought experiment of "Hilbert's hotel" provides a striking example of the alleged paradoxes. Consider a hotel with infinitely many rooms, each labelled with a distinct natural number. It's a busy night at the hotel: every room is occupied. Then another guest shows up. Fortunately, the receptionist has a brilliant idea. What if the guest in room one moves to room two, the guest in room two moves to room three, and so on? Then every current guest will have a room to herself, while room one has been made available for the new guest. This is surprising! By reassigning rooms in an entirely full hotel, we can free up a room! Obviously, no such thing is possible in an ordinary hotel with "only" finitely many rooms.

The example can be continued further, as realised already by the medievals. Once again the hotel is fully booked. But this time, *infinitely* many new guests arrive and request rooms. After a brief panic, the receptionist has another brilliant idea. What if every current guest moves to the room whose number is twice that of her present room? Then every the current guest will have a room to herself – whose room number is even – while the infinitely many rooms whose number is odd have been made available for the new guests. Thus, by reassigning rooms in an entirely full hotel, we can free up just as many rooms as were previously occupied!

What exactly is it that makes these examples so puzzling? The first example shows that there are precisely as many rooms numbered two and above as there are rooms in total. Yet the latter collection exceeds the former by one member, namely room number one. The second rearrangement shows that there are precisely as many rooms with an even number as there are rooms in total. Yet the latter collection seems twice as large as the former. So both examples show that a proper subset of the rooms can have as many members as the set of all the rooms. A proper part is thus shown to be as large as the whole! But this contradicts a seemingly solid axiom due to Euclid, which says that every whole is larger than its parts. Let's call this *Euclid's principle*. What is puzzling about the examples is that an actual infinity, such as Hilbert's hotel, would falsify Euclid's principle.

This used to be regarded as a *reductio ad absurdum* of the idea of an actual infinity. But this response was too quick. All that the examples reveal is a conflict between two of our central beliefs about the infinite, namely Euclid's principle that every whole is larger than its parts and the principle that size or number is a matter of one-to-one correspondence and thus preserved under any rearrangement of a collection. Gregory of Rimini analysed this conflict already in the fourteenth century, arguing that we are operating with two incompatible concepts of size.

My own view (which arguably is implicit in current mathematical practice) is different. I believe it is essential to our conception of number that the number of objects in a collection is an intrinsic property of the collection and thus invariant under any rearrangement of it.

Aristotle held that there are no actual infinities

For instance, the number of people initially in Hilbert's hotel is identical with the number of people in the hotel after a reassignment of rooms (and before admitting any new guests). Although each has moved to a different room, we are, after all, talking about the very same people.

Our attachment to Euclid's principle is based on an overgeneralisation from ordinary finite cases, where the principle holds. But the principle fails in extraordinary cases involving infinite totalities. So we are free to retain the connection between our concept of number and one-to-one correspondences. The lesson of the above examples is not that the concept of an actual infinity is paradoxical but that Euclid was wrong in thinking that the whole is always larger than its proper parts. This conclusion was first clearly articulated by the father of the modern theory of sets and infinity, Georg Cantor (1845–1918), and has since become part of mainstream mathematical practice. As a result of this first – and very sensible – conceptual revolution, we now possess a clear and univocal concept of size or number.

The road is now open to the development of the modern mathematics of infinity. This is a story in which Cantor is our hero. One of his most famous discoveries is a theorem now bearing his name, which says that there are more subsets of a given set than there are elements. In particular, there are more subsets of the natural numbers than there are natural numbers. This in turn means that the number of subsets of the natural numbers is larger than the number of natural numbers, that is, that one infinite number is larger than another! In fact, Cantor's theorem implies that there is an unbounded sequence of larger and larger infinite numbers.

Given this development, what becomes of the traditional concept of infinity as unboundedness? Recall that the distinction bounded versus unbounded is relative to a system of measuring sticks that can serve as bounds. An object may be too large to be bounded by one collection of measuring sticks yet be bounded by a larger kind of measuring stick. This observation becomes particularly important in light of Cantor's discoveries. The familiar collection of measuring sticks provided by the natural numbers is now supplemented with the much larger measuring sticks provided by Cantor's new numbers. In light of this, what is the more natural continuation of the traditional concept of infinity?

One option is to regard a magnitude as infinite if, and only if, it cannot be bounded by any of our *ordinary* measuring sticks, namely the natural numbers. This is the option adopted in standard mathematical practice. The resulting picture is the one described above: the natural numbers are followed by an unbounded sequence of infinite numbers.

Another option would have been to regard something as infinite just in case it cannot be bounded by *any* measuring stick, including Cantor's new and extraordinary ones. In my view, this would have been the more appropriate development. Since "finite" originally meant "bounded" or "measurable", the result of discovering a new system of longer measuring sticks should be to regard more things as finite – albeit in a generalised sense. On this alternative conceptual development, Cantor's new numbers would have been regarded not as infinite but as a generalisation of the finite. By contrast, on the conceptual development that actually prevailed, the word "infinite" shifted its meaning from something unbounded or unlimited to something that isn't bounded *by any natural number*. This conceptual revolution lacked the justification enjoyed by the other one in which Euclid's principle was rejected as inessential to our concept of number. However, despite my misgivings about this revolution, I shall continue to use what has become standard terminology and thus classify as infinite anything that cannot be bounded by a natural number. Following Cantor, we may characterise as "absolutely infinite" anything that has no bound whatsoever.

We have distinguished and clarified various conceptions of the infinite. Whether the infinite should be said to exist will obviously depend on which conception is brought to bear.

Let's begin with today's standard mathematical concept of the infinite, namely something whose size exceeds any individual natural number. Modern mathematics provides many examples of objects falling under this concept. For instance, the natural numbers are generally thought to form a set, which is infinite in the relevant sense. Moreover, since the members of this set are regarded as "all there", this infinity can be classified as actual and not merely potential. The alleged paradoxes of the actual infinite pose no threat to this view, or so I have argued.

It might be objected that these mathematical examples of the infinite don't exist in the robust sense in which planets and stars exist but merely as intellectual constructs. If true, this would mean that there are only as many truths about the infinite as follow logically from our concepts. And this might have dramatic consequences for mathematics. For some very basic set theoretic questions *provably* cannot be answered on the basis of our best current set theory. A famous example is Cantor's Continuum Hypothesis, which says that the number of real numbers is the smallest infinite number greater than the number of natural numbers. Might there simply be no answer to this question despite its apparent meaningfulness?

One way to assuage such worries is to find new mathematical axioms which may settle the question. In fact, this quest for new axioms is one of the main projects in contemporary set theory, with Hugh Woodin as its unofficial leader.

Another response would be to find realisations of the infinite in nature. For instance, if a physical line has precisely the structure of the real numbers, then there must be some objective fact as to how many real numbers there are and thus an objective truth concerning the Continuum Hypothesis.

So *is* the infinite realised in nature? Cantor thought so. Given the existence of the infinite in mathematics, he argued, it would conflict with the greatness and omnipotence of God if He did not also ensure that the concept is realised in nature. Needless to say, this argument relies on questionable theological premises. Hilbert took the opposite view: "The infinite is not to be found anywhere in reality, no matter what experiences and observations or what kind of science we might adduce". So who is right, Cantor or Hilbert? Current physical theory provides no clear verdict. It is of course true that space and time are without boundaries. But we have seen that this does not entail their infinitude. There are other possible realisations of the infinite as well, in connection with black holes or quantum fields, the theories of which are up to their necks in infinities – if taken literally. However, it is far from clear that every aspect of these theories *should* be taken literally.

Finally, there is the question of the existence of the absolute infinite, that is, a magnitude that transcends any bound whatsoever. Theology apart, there is no reason to expect such infinities to be found outside of the world of mathematics. Apparent mathematical examples include the collection of all sets and the collection of all numbers (finite and transfinite). But here paradox is lurking. As we've seen, the concept of the absolute infinite is intimately related with the ancient concept of infinity. Serious worries therefore arise about the coherence of the *actual* existence of such infinities as opposed to their merely *potential* existence as an unbounded process of generation of ever larger sets and numbers.

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This article is based on a lecture given at the Big Ideas pub philosophy group. You can listen to the original event and find upcoming events at bigi.org.uk.