# Choice sequences: a modal and classical analysis

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# Introduction

The ancient idea of **potential infinity**: more and more objects are generated, but the generation is never completed; we never have all of the objects simultaneously available.

Brouwer contributed an essentially new idea: **potentially infinite objects** and a theory of them.

**We wish to explain choice sequences to "classical minds"**

- we use classical logic in a language with one or more sets of modal operators;
- we define key concepts of intuitionistic real analysis and recover central results, using appropriate translations from the non-modal language of real analysis into our modal language;
- in particular, we establish the continuity theorems, which are distinctive of intuitionistic analysis.

# Lawlike vs. lawless sequences

The decimal expansion of $\pi$ is a **lawlike sequence**: it can be effectively generated in a deterministic manner. By contrast, Brouwer's **lawless sequences** involve genuine indeterminacy.

Suppose $\alpha(3)$ is not yet determined. So it is possible for $\alpha(3)$ to be 0 and also possible for $\alpha(3)$ to be 1. But if we realize the former possibility, then the latter possibility is no longer available.

As we'll see, lawless sequences give rise to **strong counterexamples**, i.e. theorems which from a classical point of view are false.

# The modality of determinateness

**A useful heuristic: a metaphysically open future**

- Necessity (or "determinateness") is a matter of being made true by the facts currently available
- Possibility (or "openness") is a matter of not being ruled out by the facts that are currently available (in the sense that the negation is not made true by the facts currently available)

This modality is naturally taken to be subject to S4 but nothing stronger: got reflexivity and transitivity, but not symmetry, directedness or the like.

# Connecting intuitionistic logic and S4

The Gödel translation † is given by the following clauses:

$$
\begin{aligned}
\phi &\mapsto \Box\phi \qquad \text{for } \phi \text{ atomic}\\
\neg\phi &\mapsto \Box\neg\phi^\dagger\\
\phi \vee \psi &\mapsto \phi^\dagger \vee \psi^\dagger\\
\phi \wedge \psi &\mapsto \phi^\dagger \wedge \psi^\dagger\\
\phi \rightarrow \psi &\mapsto \Box(\phi^\dagger \rightarrow \psi^\dagger)\\
\forall x\, \phi &\mapsto \Box\forall x\, \phi^\dagger\\
\exists x\, \phi &\mapsto \exists x\, \phi^\dagger
\end{aligned}
$$

The translation can be extended to plural or second-order logic in an analogous way.

# Connecting intuitionistic logic and S4

Let $\vdash_{S4}$ be deducibility in the logic that results from combining S4 with classical first-order logic. Then we have the following theorem.

**Theorem**

*Let $\vdash_{int}$ be intuitionistic deducibility in the given language. Let $\vdash_{S4}$ be the corresponding deducibility relation in S4 and classical logic. Then we have:*

$$\phi_1, \ldots, \phi_n \vdash_{int} \psi \quad \textit{iff} \quad \phi_1^\dagger, \ldots, \phi_n^\dagger \vdash_{S4} \psi^\dagger.$$

# Lawless sequences

Our comparison will be Troelstra. He uses intuitionistic logic, while we use classical S4; but we have the mentioned theorem.

For convenience, we use negative free logic. (But we could, if desirable, tweak our approach to avoid this.) So '$t = t$' can be used as an existence predicate.

We require positive stability of identity as well as an existentially qualified form of negative stability:

$$t = t' \to \Box(t = t') \tag{1}$$

$$t \neq t' \wedge t = t \wedge t' = t' \to \Box(t \neq t') \tag{2}$$

# Lawless sequences

We use $\alpha$, $\beta$, to range over sequences, and *a*, *b*, etc. to range over finite sequences (coded within arithmetic).

The finite sequences are extensional objects, which have their values of necessity wherever the sequences exist.

The choice sequences are intensional objects: they are not exhaustively characterized by their initial segments, for potentially they get more values.

Write $\alpha(m)\downarrow$ and and $\alpha(m)\uparrow$ for $\exists n(\alpha(m) = n)$ and its negation, respectively. We pronounce these as "$\alpha(m)$ is defined" and "$\alpha(m)$ is undefined", respectively.

# Lawless sequences

Notice that we get positive and (a conditional form of) negative stability of the values of a choice sequence:

$$\alpha(m) = n \rightarrow \Box\alpha(m) = n \tag{3}$$

We require that the values of a choice sequence be defined sequentially; that is, that there be no gaps:

$$\Box\forall\alpha\forall m, n(m \leq n \wedge \alpha(n)\downarrow \rightarrow \ \alpha(m)\downarrow) \tag{4}$$

# Lawless sequences

Let $is(\alpha) = a$ formalize that $a$ is the initial segment of $\alpha$ that has been defined, i.e. $a = \langle \alpha(0), \ldots, \alpha(n-1) \rangle$ where $\alpha(n)\uparrow$. Notice that $is(\alpha) = a$ is neither positively or negatively stable.

Our next axiom states that necessarily there is an empty sequence:

$$\Box \exists \alpha \, \alpha(0) \uparrow \tag{5}$$

Of course, like every other sequence, it is possible for an empty sequence to pick up values in the future.

# Lawless sequences

We also have an axiom stating that for every choice sequence $\alpha$ and finite sequence $b$, it is possible for $\alpha$ to continue beyond its initial segment as described by $b$:

$$\Box\forall\alpha\forall a(is(\alpha) = a \rightarrow \Box\forall b(\Diamond is(\alpha) = a^\frown b)) \tag{6}$$

(Of course, the analogous claim would be false of lawlike sequences.)

We can now derive (the modal translation of) Troelstra's axiom LS1, which says that for every initial sequence, there is a sequence beginning in that way.

# Compossibility of choice sequences

Our next axiom says something about when possibilities are compossible (that is, jointly possible). Consider two distinct $\alpha$ and $\beta$. Suppose $\alpha$ could continue in one way and that $\beta$ could continue in some way. Then it's possible for both sequences to continue as described.

**Some notation**

Let $\neq (\alpha, \beta_1, \ldots, \beta_n)$ abbreviate $\alpha \neq \beta_1 \wedge \ldots \wedge \alpha \neq \beta_n$.

Let $\#(\alpha_0, \ldots, \alpha_n)$ formalizes the claim that $\alpha_i \neq \alpha_j$ for each $0 \leq i < j \leq n$.

# Compossibility of choice sequences

We adopt an axiom scheme stating that possibilities concerning distinct choice sequences are compossible:

$$\#(\alpha_0, \ldots, \alpha_n) \wedge \Diamond\phi_0(\alpha_0) \wedge \ldots \wedge \Diamond\phi_n(\alpha_n) \rightarrow \Diamond(\phi_0(\alpha_0) \wedge \ldots \wedge \phi_n(\alpha_n)) \quad (7)$$

where each $\phi_i$ has only $\alpha_i$ as a parameter, no other choice sequence.

For example, if it is possible for the next entry of $\alpha$ to be 0 and it is possible for the next entry of $\beta$ to be 1, then it is possible for both of these entries simultaneously to be as described.

Let $\mathit{CoExt}(\alpha, \beta)$ state that $\alpha$ and $\beta$ are coextensive: $\Box\forall n(\alpha(n) = \beta(n))$.

## Proposition

*We can derive the Gödel translation of Troelstra's LS2:*

$$\mathit{Coext}(\alpha, \beta) \vee \neg \mathit{Coext}(\alpha, \beta) \tag{8}$$

*Proof*. Suppose $\alpha = \beta$. Then the first disjunct is true. So suppose instead that $\alpha \neq \beta$. Then it is possible that $\alpha$ should continue in one way and also possible that $\beta$ should continue in an extensionally different way. So the two developments are jointly possible. Thus, we get

$$\Box\neg\Box\forall n(\alpha(n) = \beta(n)) \tag{9}$$

which is the modal translation of the second disjunction. ⊣

# The “open data” axiom

An easy version of this axiom is:

$$A(\alpha) \rightarrow \exists n(\alpha \in n \wedge \forall \beta \in n A(\beta)) \tag{10}$$

That is, if we make the judgment $A(\alpha)$, then this is based on only finitely much information about $\alpha$, that is, on some finite initial sequence coded by $n$ (thus ‘$\alpha \in n$’).

The following axiom scheme makes sense in our setting:

$$\phi(\alpha, \beta_1, \ldots, \beta_n) \wedge \neq (\alpha, \beta_1, \ldots, \beta_n) \wedge is(\alpha) = a \rightarrow \\ \Box(\forall \gamma \in a)(\neq (\gamma, \beta_1, \ldots, \beta_n) \rightarrow \phi(\gamma, \beta_1, \ldots, \beta_n)) \tag{11}$$

That is, $\phi$ only “looks at” the initial segment of $\alpha$ that is available at the relevant world. Whatever $\phi$ says about $\alpha$, it also says about any $\gamma$ that shares the mentioned initial segment.

# The "open data" axiom

Our axiom (11) entails (the modal translation of) Troelstra's axiom open data axiom, LS3.

Troelstra goes on to introduce some further axioms, which we do not discuss (though perhaps later).

# Generation and its modality

Let $\alpha$ be a lawlike sequence. Suppose that only its first *n* values have been computed. Then $\alpha(n)$ hasn't yet been explicitly determined, although it is implicitly determined by the law governing the sequence. To represent this, we need a second pair of modal operators.

The previous modal operators represent *D*-**modality** ("being determined").

We add modal operators for *G*-**modality** ("generating (in a deterministic manner)"). In terms of our heuristic of a metaphysically open future:

- *G*-possibility: we can make true by carrying out suitable deterministic computations or constructions
- *G*-necessity: will be true whatever deterministic computations or constructions we carry out

*G*-modality is governed by S4.2 for the kinds of reasons articulated elsewhere.

The fact that generation is deterministic is nicely brought out in the following lemma.

**Lemma**

*Suppose $\phi$ and $\psi$ are positively G-stable. Suppose each formula is G-possible. Then the conjunction of the two formulas is also G-possible.*

This result crucially depends on the G (or .2) axiom. It is easy to produce Kripke models that show that the result does not hold in S4 and thus in particular not for our *D*-modality (as shown by our example on s. 4).

# Combining the two modalities

We have two pairs of modal operators for $D$-modality and $G$-modality. The two modalities were used to explicate lawless sequences and lawlike ones, respectively. How are the two modalities related?

The question is obligatory because we want to handle both lawlike and lawless sequence (plus things in between) in a unified way, where the various sorts of sequence cohabit a single domain.

- We want to take the sum of two real numbers, say a lawless one and a lawlike one.
- To say that there are both rational and irrational numbers, we need not only lawless sequences but also lawlike ones.

# Defining ‘lawlike’

We can now define what it is for $\alpha$ to be lawlike, namely, for it to be *D*-necessary that we can generate a proper extension of it.

$$\Box_D \forall a(is(\alpha) = a \rightarrow \Diamond_G \exists b(is(\alpha) = b \wedge a < b)$$

This is something that probably Troestra or the ordinary non-modal approach cannot even express.

# Real number generators

How can a choice sequence $\alpha$ be seen as encoding information about a real number? There are many possible approaches.

One good one is to use Cauchy sequences: $\alpha(n)$ encodes a rational, and the sequence is subject to the constraint that the sequence be Cauchy.

# Functions on choice sequences

Let $R$ be a relation between $\alpha$ (a choice sequence) and $b$ (some finite sequence).

The idea is that, as more and more information about $\alpha$ becomes available, $R$ will relate $\alpha$ to longer and longer finite sequences, which provide better and better approximations of the value of a function (represented by $R$) on $\alpha$ as argument.

We require positive but not negative $D$-stability of atomic statement involving $R$:

$$R(\alpha, b) \to \Box_D R(\alpha, b)$$

# Functions on choice sequences

Additionally, we require that when we have an approximation $b$ for the value with of the function applied to a sequence $\alpha$, then it is possible to extend the approximation of the value by means of a suitably large extension of the approximation of the argument sequence.

More precisely, we can generate a natural number $N$ such that it's determined that, if the approximation of the argument is extended by at least $N$ entries, then the approximation of the value can be extended by at least one entry:

$$R(\alpha, b) \wedge is(\alpha) = a \rightarrow \Diamond_G \exists N \Box_D (is(\alpha) \text{ is } N \text{ longer than } a \rightarrow \Diamond_G \exists n R(\alpha, b^\frown n)) \quad (12)$$

# Liberal vs. strict potentialism

(Linnebo and Shapiro, 2019) distinguish two qualitatively different forms of potentialism:

**Liberal potentialism**: Mathematical objects are generated successively in an incompletable process of generation. But we take a realist attitude towards the modality; in particular, a modal truth can be true in virtue of the entire space of possibilities.

**Strict potentialism**: Not only are mathematical objects generated successively, every truth is "'made true", or "fully accounted for", at some stage of the incompletable process of generation.

# Liberal vs strict: consequences concerning functions

A liberal potentialist quantifying **only over lawlike sequences** can define

$$F(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is rational;} \\ 1 & \text{if } \alpha \text{ is irrational.} \end{cases} \tag{13}$$

This is an extreme case of the function "looking ahead" at the argument beyond its initial segment.

What about a *strict* potentialist who quantifies over lawlike sequences only? Presumably there is no "looking ahead", since $G$-modal facts are not assumed to be determinate.

Suppose we quantify **only over lawless sequences**. Then, regardless of whether we are liberal or strict, it is not an option to "look ahead" in this way (by the "open data" requirement).

# Liberal vs strict: consequences concerning functions

The most interesting case is where we quantify over both lawless and lawlike sequences, as well as everything in between.

We need to ensure

$$\Box_D \forall\alpha\forall\beta(\alpha \sim \beta \rightarrow F(\alpha) \sim F(\beta)) \tag{14}$$

where $\alpha \sim \beta$ means that $\alpha$ and $\beta$ are equivalent real number generators.

A well-definedness requirement on $F$:

$$\forall N \exists M (\mathit{is}(\alpha) = \mathit{is}(\beta) \text{ and of length} > M \rightarrow \\ \mathit{is}(F(\alpha)) = \mathit{is}(F(\beta)) \text{ and of length} > N) \tag{15}$$

# Liberal vs strict: consequences concerning functions

Then:

*Liberal + well-definedness requirement*

$\Rightarrow$ *can't take advantage of full knowledge of G-facts*

$\Rightarrow$ *continuity.*

Linnebo, Ø. and Shapiro, S. (2019).
Actual and potential infinity.
*Noûs*, 53(1):160–191.