Two kinds of potential domains: some logical and historical remarks

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Weyl on potential infinity

According to the great mathematician Hermann Weyl, " 'inexhaustibility' is essential to the infinite" (Weyl, 1918, 23).

In *Levels of Infinity* (Weyl, 1930, 19), we read:

> *the sequence of all possible numbers arising through a process of generation in accord with the principle that from a given number n, there can always be generated a new one, the next number, n'. Here the existent is projected onto the background of the possible, of an ordered manifold of possibilities producible according to a fixed procedure and open to infinity.*

Actualism and potentialism

Actualism: There is no use for modal notions in mathematics, whether explicit or implicit.

Potentialism: Yes, there is such a use. For some mathematical objects are generated successively in such a way that it is impossible to complete the process of generation.

Domains that are generated successively and cannot be completed are said to be **incompletable**.

For Weyl, every infinite domain is incompletable—but **there are two kinds of such domains**!

Our aims

1. to connect Weyl on FOM with a recent distinction between liberal and strict potentialism (Linnebo and Shapiro, 2019), thus allowing the historical and the contemporary debates to inform each other.
2. to clarify how quantification over an incompletable domain can and should be understood.
3. to understand Weyl's novel distinction between two kinds of incompletable domains.

Two kinds of incompletable domains

Weyl distinguishes between two importantly different kinds of incompletable domains: those that are extensionally determinate (ED) and those that are not.

This yields a three-way classification of kinds of domains:

- completable (with actual or completed as a sub-kind)
- incompletable but extensionally determinate
- incompletable and not extensionally determinate

Extensional determinacy

A concept's being "clearly and unambiguously defined"

> *does not imply that this concept is extensionally determinate, i.e., that it is meaningful to speak of the existent objects falling under it as an ideally closed aggregate which is intrinsically determined and demarcated. (Weyl, 1919, 109)*

What is the logical "cash value" of these philosophical ideas?

> *Suppose P is a property pertinent to the objects falling under a concept C. [...] if the concept C is extensionally determinate, then not only the question "Does a have the property P?" [...] but also the existential question "Is there an object falling under C which has the property P?", possesses a sense which is intrinsically clear. (ibid.)*

Not only 'Pa' but also '$(\exists x : C)Px$' has an "intrinsically clear" sense, i.e. LEM holds.

The intuition of iteration assures us that the concept "natural number" is extensionally determinate. [...] However, the universal concept "object" is not extensionally determinate—nor is the concept "property," nor even just "property of natural number". (Weyl, 1919, 110)

An example of an ED domain

The natural numbers as paradigmatic example of an ED domain:

- generated by 1 and successor;
- with mathematical induction as closure.

> *The intuition of iteration assures us that the concept "natural number" is extensionally determinate. (Weyl, 1919, 110)*

As the domain of the natural numbers is the extension of the concept "natural number", it is extensionally determinate.

Weyl's "mathematical process"—generating ED domains in analysis

In *Das Kontinuum* (Weyl, 1918) Weyl describes a process of generation of extensionally determinate domains.

- Start from the natural numbers (generated from 1 and the primitive relation of successor, with mathematical induction).
- Use the standard logical operations to obtain complex judgements expressing complex properties of the natural numbers.
- Crucial requirement: **quantification** is only allowed to range **over the natural numbers** (to avoid vicious circularity).
- **ED sets** are then the **extensions of the resulting complex properties**, *modulo extensionality*.
- Iterate the process?

Weyl against the combinatorial conception of set

Like Poincaré, Weyl rejects the combinatorial conception of set as applied to infinite domains.

> *The notion of an infinite set as a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and surveyed as a whole by consciousness, is nonsensical: "inexhaustibility" is essential to the infinite. (Weyl, 1918, 23)*

As an infinite set is incompletable, to describe it one needs a rule that "indicates properties which apply to the elements of the set and to no other objects".

Non-ED domains according to Weyl

Recall:

> *The intuition of iteration assures us that the concept "natural number" is extensionally determinate. [...] However, the universal concept "object" is not extensionally determinate—nor is the concept "property," nor even just "property of natural number". (Weyl, 1919, 110)*

A set of natural numbers is the extension of a property of the natural numbers. Since "property of the natural numbers" is not extensionally determinate, also the powerset of the natural numbers is not extensionally determinate.

In fact, for Weyl the collection of all real numbers is not extensionally determinate.

A non-ED domain in constructive type theory

To introduce his "reflecting universes" in type theory Martin-Löf writes:

> *Recall that there can be no set of all sets, because we are not able to exhibit once and for all all possible set forming operations. (The set of all sets would have to be defined by prescribing how to form its canonical elements, i.e. sets. But this is impossible, since we can always perfectly well describe new sets, for instance, the set of all sets itself.) (Martin-Löf 1984, p. 87)*

Similar ideas are found in (Tait, 1998) and (Studd, 2019, §7.5).

How to generalize over an incompletable domain

Generalizations over a completable domain can be understood in an **instance-based** manner: i.e. $\forall x \, \varphi(x)$ is true because each and every object a in the domain is such that $\varphi(a)$.

How, though, should generalizations over an incompletable domain be understood? It is far from clear that one is then entitled to an instance-based understanding.

We will now look at a series of proposals, ordered from the less to the more ambitious.

(a) Restricting to ED domains

Weyl (1921) justifies his use of classical logic in *Das Kontinuum* (1918) as follows:

> *If I run through the sequence of numbers and terminate if I find a number of property* **E***, then this termination will either occur at some point, or it will not; that is, it is so, or it is not so, without any wavering and without a third possibility. (Weyl, 1921, 97)*

The restriction to ED domains is essential. The usual impredicative proof that the reals have the LUB property illicitly assumes

> *that the totality of "all possible" properties is in itself determined and delimited, that is, in principle surveyable (ibid., 88).*

(b) Schematic generality

Hilbert says of "$a + b = b + a$" that it

> *is in no wise an immediate communication of something signified but is rather a certain formal structure whose relation to the old finitary statements*
>
> $$2 + 3 = 3 + 2$$
> $$5 + 7 = 7 + 5$$
>
> *consists in the fact that, when a and b are replaced in the formula by the numerical symbols 2, 3, 5, 7, the individual finitary statements are thereby obtained, i.e., by a proof procedure, albeit a very simple one. (Hilbert, 1926, 196)*

Compare (Parsons, 2006) and (Glanzberg, 2004) in the case of set-theoretic potentialism.

Advantages of schematic generality:

- we retain classical logic;
- this conception works whether or not the domain is ED.

Disadvantage:

- very limited ability to generalize, we have only Π_1-generalizations. As Hilbert puts it, there are statements that "from our finitary perspective [are] *incapable of negation*" (194).

(c) Weyl (1921)'s alternative

Is there a natural number that has some decidable property P? Weyl wrote:

> *Only the finding that has actually occurred of a determinate number with the property P can give a justification for the answer "Yes," and—since I cannot run a test through all numbers—only the insight, that it lies in the essence of number to have the property not-P, can give a justification for the answer "No"; Even for God no other ground for decision is available. (Weyl, 1921, p. 97)*

Thus, both the quantifiers have an interpretation that enables a quantified statement to be accounted for at a stage of a generative process solely on the basis of material that is available at that stage. Thus, it doesn't matter if the domain extends beyond that stage in a way that isn't ED.

As we'll see, however, the price is that we must use intuitionistic logic.

Summary

conception of generality	how much absolute generality	abs. gen. available for which domains	logic validated
restricting to ED	no abs. gen.	none	classical
schematic	Π_1 only	all domains	classical
Weyl (1921)	full	all domains	intuitionistic

Liberal vs. strict potentialism

(Linnebo and Shapiro, 2019) distinguish two qualitatively different forms of potentialism:

Liberal potentialism: Mathematical objects are generated successively in an incompletable process of generation. But we take a realist attitude towards the modality; in particular, a modal truth can be true in virtue of the entire space of possibilities.

Strict potentialism: Not only are mathematical objects generated successively, every truth is "'made true", or "fully accounted for", at some stage of the incompletable process of generation.

We believe the historical and the contemporary debates can inform each other:

1. Weyl's writings from this period shed light on when each type of potentialism is appropriate
2. the liberal/strict distinction sheds light on an important shift in Weyl's view from 1918 to 1921

When is liberal potentialism permissible?

Re. (1): liberal potentialism is permissible iff the potential domain is ED.

- When a potential domain is ED, this justifies a realist attitude towards the modality, and it is possible for a modal statement to be true solely in virtue of the entire space of possibilities.
- When a potential domain isn't ED, it doesn't make sense for a statement to be true solely in virtue of the entire space of possibilities.
- In this latter case, we must be **strict potentialists** and use Weyl (1921)'s **non-instance-based generality** (perhaps supplemented with schematic generality).

Weyl's move from liberal to strict potentialism

Now for point (2): Can we connect liberal vs. strict potentialism with classic discussions in foundations of mathematics?

(Weyl, 1918) (and *classical* predicativism more generally): since the collection of sets of natural numbers is not ED, liberal potentialism is not an option. Thus, we must either restrict to ED domains or use schematic generality . . . Or?

Weyl (1921) discovers another option: non-instance-based generality.

There is a trade-off between the two options available in 1918 and the new 1921 option:

- restricting to ED domains or using schematic generality are superior wrt. the strength of one's logic of quantification.
- Weyl (1921)'s generality is superior wrt. expressive power.

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Brouwer–the revolution!

Inspired by Brouwer, (Weyl, 1921, 88-89) enthusiastically embraces non-instance-based generality instead of restricting to ED domains or using schematic generality.

> *So I now abandon my own attempt and join Brouwer. I tried to find solid ground in the impending dissolution of the State of analysis [. . .] without forsaking the order upon which it is founded, by carrying out its fundamental principle purely and honestly. I believe I was successful—as far as this is possible. For this order is in itself untenable, as I have now convinced myself, and Brouwer—that is the revolution!*

Our acct. of (2) summarized: Weyl's shift from 1918 to 1921 crucially involved a shift from liberal to strict potentialism concerning the natural numbers.

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