Axiomatizing grounded truth?

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Content

1 Different conceptions of truth and groundedness
   - Kripke
   - Leitgeb on grounded truth
   - A semantic construction for truth and groundedness

2 Axiomatizing a semantic theory of truth
   - Criteria of adequacy
   - $\omega$-categoricity

3 Prospects for the different versions
   - Kripke axiomatized?
   - Leitgeb axiomatized?
   - An axiomatic theory of truth and groundedness?
Kripke on groundedness I

(1) Most (i.e, a majority) of Nixon’s assertions about watergate are false.

In general, if a sentence such as (1) asserts that (all, some, most, etc.) of the sentences of a certain class C are true, its truth value can be ascertained if the truth values of the sentences in the class C are ascertained. If some of these sentences themselves involve the notion of truth, their truth value in turn must be ascertained by looking at other sentences, and so on. If ultimately this process terminates in sentences not mentioning the concept of truth, so that the truth value of the original statement can be ascertained, we call the original sentence grounded; otherwise, ungrounded. (Kripke [1975, p.693f.])
We wish to capture an intuition of somewhat the following kind. Suppose we are explaining the word 'true' to someone who does not yet understand it. We may say that we are entitled to assert (or deny) of any sentence that it is true precisely under the circumstances when we can assert (or deny) the sentence itself. [...] our suggestion is that the "grounded" sentences can be characterized as those which eventually get a truth value in this process. (Kripke [1975, p701.])

Kripke has two informal characterizations of groundedness: top-down, bottom-up.
Fixed-points I

Definition (Kripke Jump)

A Kripke jump $\Gamma^+ : P(\mathcal{L}_T)^2 \to P(\mathcal{L}_T)^2$ is defined as follows:

$\Gamma^+(E, A) := (\{\#\varphi : (E, A) \models_+ \varphi\}, \{\#\varphi : (E, A) \models_+ \neg \varphi\})$, for some evaluation scheme $+$, such as WK, SK or SV.

Definition (Fixed-point)

Let $E, A \subseteq \mathcal{L}_T$. Then $(E, A)$ is a fixed-point of $+$ iff $\Gamma^+(E, A) = (E, A)$.

Remark: The arithmetical vocabulary is interpreted by the standard model.
Fixed-points II

Kripke’s construction is very rich and allows for variations. Kripke is neutral regarding:

- What is the correct evaluation scheme?
- Which fixed-point gives the correct extension of truth?
- Closed off fixed points?
- Minimal versus all fixed-points?
Minimal fixed-point

We get the minimal fixed point $I_{\Gamma_+}$ by a bottom-up construction:

\[
\Gamma^+_0 = (\emptyset, \emptyset);
\]

\[
\Gamma^+_{\alpha+1} = \Gamma^+(\Gamma^+_\alpha);
\]

\[
\Gamma^+_\beta = \left( \bigcup_{\alpha<\beta} \Gamma^E_\alpha, \bigcup_{\alpha<\beta} \Gamma^A_\alpha \right);
\]

Groundedness as truth or falsity in the minimal fixed-point.

Definition (Kripke)

A sentence $\varphi$ is grounded iff $\varphi$ is in the extension or anti-extension of the minimal fixed-point.
Dependence

Yablo identifies two aspects of groundedness:

- inheritance, bottom-up.
- dependence, top-down.

*Now Kripke’s Theory is very instructive about the inheritance aspect of semantic grounding, but it really does little to supply the dependence aspect of our intuition. (Yablo [1982, p.118])*
Dependence

**Definition**

φ depends on Φ iff for all Ψ ⊆ 𝒟, Ψ |= φ ⇔ Ψ ∩ Φ |= φ.

The dependence operator $D^{-1} : P(𝒟) \rightarrow P(𝒟)$ is defined as follows:

$D^{-1}(Φ) := \{ #φ : φ \text{ depends on } Φ \}.$

$D^{-1}$ has a minimal fixed-point $I_{D^{-1}}$.

**Grounded**

A sentence φ is **grounded** iff it depends on nonsemantic states of affairs iff $φ \in I_{D^{-1}}$. 
Grounded Truth

Truth for sentences that depend on nonsemantic states of affairs:

\[ \Phi_0 = \emptyset; \]
\[ \Phi_{\alpha + 1} = D^{-1}(\alpha); \]
\[ \Phi_\beta = \bigcup_{\alpha < \beta} \Phi_\alpha; \]

\[ \Lambda_0 = \emptyset; \]
\[ \Lambda_{\alpha + 1} = \{ \varphi : \varphi \in \Phi_{\alpha + 1} \& \Lambda_\alpha \models \varphi \}; \]
\[ \Lambda_\beta = \bigcup_{\alpha < \beta} \Lambda_\alpha; \text{ for limit } \beta \]

\( I_\Lambda \) is the minimal fixed-point of \( \Lambda \).
Grounded truth via necessity

- Semantical theories of truth usually focus on one fixed-point model (Minimal fixed-point, maximally intrinsic fixed-point)
- The structure of fixed-points is rich.
- Include all the fixed-points and its structure in one model, a possible worlds model.
- Every world should represent one possible fixed-point.
- Define $\varphi$ is grounded as $\varphi$ is true in all fixed-points or $\neg \varphi$ is.
- Recapture groundedness via necessary true or false.
A semantic construction for truth and groundedness

The generalization of the fixed-point construction to possible worlds models is due to Halbach/Welch and Stern.

- **Language** $\mathcal{L}^m = \mathcal{L}_A \cup \{ T, N, P \}$.
- **Frame** $F = (\mathcal{W}, R)$ and a possible worlds model $M = (F, f)$.
- $\mathcal{W}$ is a set of possible worlds, $R$ is an accessibility relation.
- $f$ is a partial evaluation function, i.e. $f : \mathcal{W} \rightarrow P(\omega)$.
- where each $w \in \mathcal{W}$ is a partial model $(\omega, (E_T, A_T), (E_N, A_N), (E_P, A_P))$. 
Consistent extensions

- Let there be a fixed-Gödel coding of $\mathcal{L}^m$, such that $\# \varphi$ is the Gödel number of $\varphi$.
- $SENT := \{ \# \varphi : \varphi \text{ is a sentence of } \mathcal{L}^m \}$.
- $NSENT := \omega \setminus SENT$.
- For $X \subseteq SENT$ we say that $X$ is consistent iff $\# \varphi \in X \Rightarrow \# \neg \varphi \notin X$.
- $CONS := \{ X \subseteq SENT : X \text{ is consistent } \}$. 
Let $f : W \rightarrow CONS$.

**Definition**

Let $F = (W, R)$ be a frame and $f$ an evaluation function.

- $E^W_T(f) := f(w)$
- $A^W_T(f) := NSENT \cup \{ \# \varphi : \# \neg \varphi \in E^W_T(f) \}$
- $E^W_N(f) := \bigcap_{wRv} E^v_T(f)$
- $A^W_N(f) := \bigcup_{wRv} A^v_T(f)$
- $E^W_P(f) := \bigcup_{wRv} E^v_T(f)$
- $A^W_P(f) := \bigcap_{wRv} A^v_T(f)$
\[ M, w \models_{SK} \varphi \]

### Definition

Let \( M \) be a model \((F, f)\), with \( F = (W, R) \) a frame and \( f \in EV \). \( M, w \models_{SK} \varphi \) is defined as follows:

(i) \( M, w \models_{SK} s = t \) iff \( s^N = t^N \);
(ii) \( M, w \models_{SK} \neg s = t \) iff \( s^N \neq t^N \);
(iii) \( M, w \models_{SK} Tt \) iff \( t^N \in E^w_T \);
(iv) \( M, w \models_{SK} \neg Tt \) iff \( t^N \in A^w_T \);
(v) \( M, w \models_{SK} Nt \) iff \( t^N \in E^w_N \);
(vi) \( M, w \models_{SK} \neg Nt \) iff \( t^N \in A^w_N \);
(vii) \( M, w \models_{SK} Pt \) iff \( t^N \in E^w_P \);
(viii) \( M, w \models_{SK} \neg Pt \) iff \( t^N \in A^w_P \);
\( M, w \models^{SK} \varphi \)

**Definition (Cont.)**

(ix) \( M, w \models^{SK} \neg \varphi \) iff \( M, w \models^{SK} \varphi \);

(x) \( M, w \models^{SK} \varphi \land \psi \) iff \( M, w \models^{SK} \varphi \) and \( M, w \models^{SK} \psi \);

(xi) \( M, w \models^{SK} \neg(\varphi \land \psi) \) iff \( M, w \models^{SK} \neg \varphi \) or \( M, w \models^{SK} \neg \psi \);

(xii) \( M, w \models^{SK} \forall x \varphi \) iff for all \( n \in \omega \), \( M, w \models^{SK} \varphi(n/x) \);

(xiii) \( M, w \models^{SK} \neg \forall x \varphi \) iff there exists a \( n \in \omega \), \( M, w \models^{SK} \neg \varphi(n/x) \);

We also use \( f, w \models^{SK} \varphi \) for \( M, w \models^{SK} \varphi \) if \( M \) is based on \( f \) and clear from the context.
Fixed-points

Definition (Visser)
Let $\mathbb{X} = (X, \leq)$ be a poset. Then $\mathbb{X}$ is a ccpo (coherent complete partial order) iff every consistent subset of $X$ has a supremum, where $Y \subseteq X$ is consistent iff for all $x, y \in Y$, $\{x, y\}$ has an upper bound in $\mathbb{X}$.

Theorem (Visser's fixed point theorem)
Let $\mathbb{X} = (X, \leq)$ be a ccpo and $\Gamma$ a monotone operator on $\mathbb{X}$, then $\mathbb{F}(X, \Gamma) = (\{x \in X : x = \Gamma(x)\}, \leq \upharpoonright \{x \in X : x = \Gamma(x)\})$ is a ccpo.
Let $EV_c$ be the set of consistent evaluation functions. For $f, g \in EV_c$ define $f \leq g :\iff$ for all $w \in W$, $f(w) \subseteq g(w)$.
Let $E := (EV_c, \leq)$.

**Lemma**

$E$ is a ccpo.
Kripke Jump

Lemma

If $f \in EV_c$, then for all $w$, \{#\varphi : f, w \models^{SK} \varphi\} is consistent.

The modal Kripke jump operator $\Delta : EV_c \to EV_c$ is defined as $\Delta(f)(w) := \{#\varphi : f, w \models^{SK} \varphi\}$.

Lemma

$\Delta$ is a monotone operator on $E$, i.e. $f \leq g \Rightarrow \Delta(f) \leq \Delta(g)$.

Corollary

There are consistent $\Delta$ fixed-points.
Let $F$ be a reflexive frame and $f$ a consistent fixed-point. Then for all $w$ the following principles hold, i.e. $f,w \models$

1. $KF + \text{consistency}$.
2. $\forall x (\text{Snt}(x) \rightarrow (N(x) \rightarrow T(x)))$
3. $\forall x (\text{Snt}(x) \rightarrow (T(N(x)) \leftrightarrow N(x)))$
4. $\forall x (\text{Snt}(x) \rightarrow (T(\neg N(x)) \leftrightarrow P(\neg x)))$
5. $\forall x (\text{Snt}(x) \rightarrow (T(P(x)) \leftrightarrow P(x)))$
6. $\forall x (\text{Snt}(x) \rightarrow (T(\neg P(x)) \leftrightarrow N(\neg x)))$
### Soundness

**Definition**

Let \( X \) be a ccpo and \( \Gamma \) a monotone operator on \( X \) and \( x \in X \).

- \( x \) is sound iff \( x \leq \Gamma(x) \).
- \( \text{SOUND}_\Gamma := \{ x \in X : x \text{ is sound} \} \).

**Lemma**

*If \( x \) is sound, then there is a fixed-point \( y \), such that \( x \leq y \).*
The possible worlds of the model

Let $\Gamma_{SK}$ be the simple Kripke jump operator for the language $\mathcal{L}_T$. Let $\text{SOUND}_{\Gamma_{SK}}$ be the set of sound and consistent sets of sentences. Let $W$ be a set of worlds indexed by the reals, i.e. $W := \{w_i : i \in \mathbb{R}\}$.

**Definition**

Let $f$ be an evaluation function $f : W \rightarrow \text{CONS}_{\mathcal{L}_m}$. $f$ is $\Gamma_{SK}$-complete iff for all $X \in \text{SOUND}_{\Gamma_{SK}}$ there is a $w_i \in W$ such that $f(w_i) = X$. 
Since $|\text{SOUND}_{\Gamma_{SK}}| = 2^\omega$ there is a bijection $h : W \rightarrow \text{SOUND}_{\Gamma_{SK}}$. Clearly $h$ is an evaluation function.

**Lemma**

- $h$ is a consistent evaluation function.
- $h$ is sound with respect to $\Delta_{SK}$.
- $h$ is $\Gamma_{SK}$-complete.
The intended model

Let $F = (W, R)$ be a frame, with $W$ as before and $R$ universal, i.e. for all $i, j \in \mathbb{R}$, $w_i R w_j$.

We construct a fixed-point model $M$ based on $h$. Let $h_0 = h$, then:

$$M_0 = (W, R, h_0)$$
$$M_{\alpha+1} = (W, R, \Delta_{SK}(h_{\alpha}))$$
$$M_{\beta} = (W, R, \bigcup_{\alpha<\beta} h_{\alpha}).$$

Let $I_h^h$ be the fixed-point (evaluation function) based on $h$. 
Groundedness defined

A sentence $\varphi$ is **grounded** iff $\varphi$ is truth determinate in all the fixed-points and has the same truth value in all fixed-points iff $\varphi$ is necessarily true or necessarily false.

**Definition**

\[ G(x) :\iff N(x) \lor N(\neg x) \]
A semantic theory of truth alone seems insufficient as an account of truth.

An axiomatic theory would complete the picture.

Axiomatic and semantic approaches can support each other.

A semantic theory can help to formulate an axiomatic theory.

Sometimes an axiomatic theory can capture a semantic theory.
Criteria

- **Soundness.** Necessary but not sufficient.
- **Similarity.**
  Examples: Tarski and $CT$ or $Kripke_{SK}$ and $KF$.
- **Proof-theoretic strength.**
  Examples: $KF$, $BKF$.
  Counterexample PUTB.
- **Completeness or categoricity?**
$\omega$-categoricity

**Definition**

Let $\Sigma$ be an axiomatic theory and $\Psi$ a class of models: $\Sigma$ is $\omega$-categorical with respect to $\Psi$ iff for all $S \subseteq L_\Sigma$:

$$(\omega, S) \models \Sigma \iff S \in \Psi.$$ 

- $\omega$-categoricity is well-motivated.
- $\omega$-categoricity is used in the literature.
Minimal Kripke fixed-point models

**Theorem (Halbach)**

*There is no $\omega$-categorical axiomatic theory of truth for the minimal fixed-point.*

Argument: For $\Sigma$ to be $\omega$-categorical with respect to the minimal fixed-point it would have to satisfy the following

$$(\omega, S) \models \Sigma \iff S = l_{\Gamma_+}.$$ 

However, satisfaction is $\Delta^1_1$ whereas $l_{\Gamma_+}$ is complete $\Pi^1_1$ for $+=SK$ as well as $SV$ based on $ca, bu$-admissibility.
Supervaluation and admissibility

Definition

Let $\Phi, \Psi \subseteq \mathcal{L}_T$.

- $\Psi$ is ca-admissible for $\Phi$ iff $\Phi \subseteq \Psi$ and $\Psi$ is consistent.
- $\Psi$ is bu-admissible for $\Phi$ iff $\Phi \subseteq \Psi$ and $\Psi \cap \neg\Phi = \emptyset$.

where $\Psi$ is consistent iff for all $\varphi$ if $\#\varphi \in \Psi$, then $\#\neg\varphi \notin \Psi$ and $\neg\Phi := \{\#\varphi : \#\neg\varphi \in \Phi\}$. 
Arbitrary fixed-points for SK

KF is an adequate axiomatization of the closed-off strong Kleene fixed-points.

**Theorem (Feferman)**

\((\omega, S) \models KF \iff S = \Gamma_{SK}(S)\).

This works because \(\Gamma_{SK}\) and \(\models_{SK}\) is \(\Delta^1_1\) (Kripke,Burgess). However KF doesn’t give us grounded truth.
Arbitrary fixed-points for SV

Theorem

There is no $\omega$-categorical axiomatization of the supervaluation fixed-points based on $ca$, $bu$-admissibility.

Argument: $\Sigma$ would have to satisfy

$$(\omega, S) \models \Sigma \iff S = \Gamma_{SV}(S).$$

However $\Gamma_{SV}$ and $\models_{SV}$ are not $\Delta^1_1$ but $\Pi^1_1$ (Burgess).

Conclusion

Kripke grounded truth is not $\omega$-categorical axiomatizable.
Minimal fixed-points for Leitgeb’s construction

**Theorem (Leitgeb)**

\( I_\Lambda \text{ is complete } \Pi_1^1. \)

**Corollary**

*There is no \( \omega \)-categorical axiomatic theory of truth for the minimal fixed-point \( I_\Lambda. \)*

Argument: For \( \Sigma \) to be \( \omega \)-categorical with respect to the minimal fixed-point it would have to satisfy the following

\[(\omega, S) \models \Sigma \iff S = I_{\Gamma^+}.\]

However, satisfaction is \( \Delta_1^1 \) whereas \( I_\Lambda \) is complete \( \Pi_1^1 \).
Conditional dependence 1

There are variations of the notion of dependence, due to Leitgeb, Meadows and Bonnay/van Vugt based on the $bu$, $ca$-admissibility conditions:

**Definition**

ϕ depends on Φ +−-conditionally on Σ iff: For all Ψ +−-admissible for Σ, ψ |= ϕ ⇔ ψ ∩ Φ |= ϕ, for + either $bu$, $ca$.

We have the conditional dependence operators

$$D_{−1}^+(Φ) := \{ \#ϕ : ϕ +−-conditionally depends on Φ \}.$$
Conditional dependence II

**Theorem (Leitgeb, Meadows, Bonnay/van Vugt)**

For all $\alpha$: $l_\Lambda^\alpha = l_\Gamma^\alpha$, where $+$ is either $ca$ or $bu$.

**Corollary**

There is no $\omega$-categorical axiomatic theory of truth for the minimal fixed-point of $l_\Lambda^+$, for $+$ = $ca$, $bu$. 

Arbitrary fixed-points for dependence

**Lemma (Leitgeb)**
\[
\Phi_\alpha = \Lambda_\alpha \cup \neg \Lambda_\alpha
\]

Similar for the conditional dependence operators: So we can define truth jump operators directly (Bonnay/van Vugt):

**Definition**

For consistent \( \Phi \subseteq \mathcal{L}_T \):

\[
\Lambda'_d(\Phi) := \{ \#\varphi : \varphi \text{ depends on } \Phi \cup \neg \Phi \quad \text{and} \quad \Phi \models \varphi \}.
\]

\[
\Lambda'_+(\Phi) := \{ \#\varphi : \varphi + \text{-conditionally depends on } \Phi \cup \neg \Phi \quad \text{and} \quad \Phi \models \varphi \}.
\]
Arbitrary fixed-points for dependence

Lemma

For consistent $\Phi \subseteq \mathcal{L}_T$: $\Lambda'_+(\Phi) = \Gamma'_+(\Phi)$

Corollary

There is no $\omega$-categorical axiomatic theory of truth for arbitrary fixed-points of $\Lambda'_+$, for $+ = ca, bu$.

The reason is that $\Lambda'_+$ and also the corresponding conditional dependence operators $D^{-1}_+$ are $\Pi^1_1$ and not $\Delta^1_1$.

Open question: Are the operators $D^{-1}$ and $\Lambda'_d \Delta^1_1$.

Conclusion

Leitgeb grounded truth is not axiomatizable (in most versions).
Axiomatizing the intended model

Axiomatizing the intended model would amount to finding an axiomatic theory $\Sigma$ such that

$$\text{for all } w \in W: f, w \models \Sigma \iff f = \mathcal{I}^h_\Delta.$$  

Conjecture: Similar to the case of minimal fixed-points, there is no $\omega$-categorical axiomatization of the intended model.
Axiomatizing arbitrary fixed-points I

Take $EV_s := \{ f : W \rightarrow \text{SOUND}_{SK} \}$
and
$E_s := (EV_s, \leq)$
and
$\Delta : EV_s \rightarrow EV_s$ is defined as $\Delta(f)(w) := \{ \#\varphi : f(w) \models^\text{SK} \varphi \}$.

- $E_s$ is a ccpo.
- $\Delta$ is monotone on $E_s$.
Axiomatizing arbitrary fixed-points II

Axiomatizing the intended model would amount to finding an axiomatic theory $\Sigma$ such that

$$\text{for all } w \in W : f \models \Sigma \iff f = \Delta(f).$$

**Lemma (Burgess)**

- $SO\text{UND}_{SK}$ is $\Delta^1_1$.
- $SO\text{UND}_{SV}$ is $\Pi^1_1$, for $bu, ca$-admissibility.

**Conjecture:** An $\omega$-categorical axiomatization of arbitrary $SK$-fixed-points should be possible.

**Question:** Is this semantic theory still a theory of grounded truth?
Conclusion

- The prospects of $\omega$-adequately axiomatizing grounded truth in the versions of Kripke and Leitgeb are not promising.
- There might be an option for the modalized version, however its status as a semantic theory of grounded truth has to be established.
- Alternative criteria of adequacy might be a way out.
Thank you!