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FROM THE AXIOM OF CHOICE TO CHOICE SEQUENCES*

The theory of choice sequences is usually considered to be far from the mainstream of mathematics. In this note we show that it did not start that way. There is a continuous development from discussions around the use of axiom of choice to Brouwer's introduction of choice sequences. We have tried to trace this development starting in 1904 and ending in 1914.[1]

In his book on choice sequences, Troelstra (1977) gives the development after 1914, but does not indicate where Brouwer got his concept. This note is a first attempt at an answer.

Our story starts in August 1904, with Zermelo writing a long letter to Hilbert, who thinks part of the letter deserves a wider audience. So he publishes it directly in *Mathematische Annalen* (Zermelo 1904).

The leisurely style is clear from the title, "Proof that every set can be well-ordered, (from a letter sent to Mr. Hilbert)", and the first sentence:

> . . . The following proof comes from conversations that I had last week with Mr. Erhard Schmidt and it is as follows.

Zermelo gave the standard argument that the axiom of choice implies the well-ordering principle. He argued that the axiom of choice was self-evident. The reactions to the proof came immediately. At the end of a note sent to *Mathematische Annalen* in December 1904, Borel states:

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*This paper has circulated as an unpublished note since 1979. It is referred to in A. S. Troelstra (1982) On the origin and development of Brouwers's concept of choice sequence, in A. S. Troelstra and D. van Dalen (eds.), *The L. E. J. Brouwer Centenary Symposium*, North-Holland, Amsterdam, and in Jan von Plato (1994) *Creating Modern Probability*, Cambridge University Press, Cambridge.

[1]Our main references are Borel (1972) and Brouwer (1975). There is a short history of choice sequences in Troelstra (1977). The translations are my own.

Nordic Journal of Philosophical Logic, Vol. 1, No. 1, pp. 95–98.
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It seems to me that the objection against it is also valid for every reasoning where one assumes an arbitrary choice made an uncountable number of times, for such reasoning does not belong in mathematics. (Borel 1972, pp. 1251–1252)

Strong words. Similar objections were made by others too. Some defended Zermelo's argument. This debate is summed up by Zermelo in *Mathematische Annalen* 1908 in "Neuer Beweis für die Möglichkeit einer Wohlordnung". For our story we note that Zermelo's paper started a heated foundational debate, with the most prominent mathematicians of the time involved.

Our story continues in France, where Borel had written that Zermelo's axiom of choice did not belong in mathematics (see quotation above). As a statesman of mathematics he gathered together the opinions of the most prominent French mathematicians of his generation—Hadamard, Baire, and Lebesgue—and had them published in *Bulletin de la Société mathématique de France* in 1905 as "Cinq lettres sur la théorie des ensembles" (Borel 1972, pp. 1253–1265). Hadamard supported Zermelo, while Baire and Lebesgue were on Borel's side. In the mathematical world these discussions were important. There were foundational problems close to mathematical practice. The two views were named formalism and intuitionism. When Brouwer first started to talk about intuitionism he showed his acceptance of the philosophy of Borel, Lebesgue, and Baire. Later he was to call them pre-intuitionists.

The debate continued in the next decade. In 1912 in "La philosophie mathématique et l'infini" (Borel 1972, pp. 2127–2136) Borel gave the following version of the debate:

It is possible to define a bounded decimal number by demanding that a thousand persons each write an arbitrary digit. One will have a well-defined number if the persons are put in line each writing in turn a digit at the end of the digits already written by those in front in the line. The disagreement starts when one tries to extend this procedure to an unbounded decimal number. I do not suppose that people dream of actually having an infinite number of persons each writing an arbitrary digit, but I believe that Mr. Zermelo and Mr. Hadamard think that it is possible to regard such a choice realized in a perfectly well-defined way even if the complete definition of the number contains an infinite number of words. For my part I think it is possible to pose problems about probability for decimal numbers which are obtained by choosing the digits either randomly or by imposing certain restrictions on the choice-restrictions leaving some randomness to the choice. But I think it is impossible to talk about one of these numbers for the reason that if one denotes it by A, two mathematicians talking about A would never be sure whether they were talking about the same number. (Borel 1972, pp. 2129–2130)

Here we have a link between the axiom of choice and the theory of choice sequences. Borel uses a number defined by a choice sequence to show the difference between the formalists and the intuitionists. Note that he no longer only considers uncountable axiom of choice. That the problems arise for all infinite uses of axiom of choice was pointed out by Lebesgue in the correspondence from 1905.

Brouwer now enters the scene from the sidelines. From the start of his mathematical career in 1907 he was interested in the formalist/intuitionist controversy. For him the foundational puzzles were in this controversy and not in the problems of the logicians. In his thesis from 1907 Brouwer comments on Zermelo's proof and agrees with Borel's critique (Brouwer 1975, p. 84). Like Borel, Brouwer believes that the problems only come with uncountable sets.

Brouwer's inaugural address from 1912, "Intuitionism and formalism", is the first place where he mentions free choices. The relevant passages are similar to the quotation from Borel above:

> Let us consider the concept: "real number between 0 and 1." For the formalist this concept is equivalent to "elementary series of digits after the decimal point", for the intuitionist it means "law for the construction of an elementary series of digits after the decimal point, built up by means of a finite number of operations." And when the formalist creates the "set of all real numbers between 0 and 1," these words are without meaning for the intuitionist, even whether one thinks of the real numbers of the formalist, determined by elementary series of freely selected digits, or of the real numbers of the intuitionist, determined by finite laws of construction. [...]
>
> If we restate the question in this form: "Is it impossible to construct infinite sets of real numbers between 0 and 1, whose power is less than that of the continuum, but greater than aleph-null?," then the answer must be in the affirmative; for the intuitionist can only construct denumerable sets of mathematical objects and if, on the basis of the intuition of the linear continuum, he admits elementary series of free selections as elements of construction, then each non-denumerable set constructed by means of it contains a subset of the power of the continuum. (Brouwer 1975, pp. 133–135)

In his book, Troelstra has given these quotations as Brouwer's first mention of choice sequences. The first quotation shows that Brouwer would only admit lawlike definition of a real number. Troelstra interprets the second quotation as showing that the intuitionists at least conceivably might use choice sequences. Another possibility is that Brouwer thought of probability statements in the same way that Borel did.

By 1914 Brouwer had changed his views on choice sequences, as pointed out by Troelstra in his book. In a review on Schoenflies and Hahn's book on set theory (Brouwer 1975, pp. 139–144) Brouwer remarks in a footnote:

Z.B. ist die Punktmenge: "alle reellen Zahlen zwischen 0 und 1 mit *Ausnahme* der endlichen Dualbrüche", nur deshalb eine wohlkonstruierte Menge, weil die duale Entwicklung einer willkürlichen Zahl dieser Menge eine Fundamentalreihe von endlichen Gruppen von gleichen Ziffern (abwechselnd 0 und l) liefert, so daß die Menge sich mittels einer Fundamentalreihe von Auswahlen unter den endlichen Zahlen bestimmen läßt. Dieser Schritt geht freilich weiter als mein römischer Vortrag,[2] und auch weiter als die *Borel*schen Ausführungen über wohlkonstruierte Mengen;[3] er erscheint mir aber als eine notwendige Konsequenz des Intuitionismus. (Brouwer 1975, p. 140)

Brouwer's lecture in Rome is from the International Congress of Mathematicians in 1908. Both Zermelo and Borel were there. Borel's paper is from 1912. In our long quotation from Borel above, this paper is mentioned in a footnote as a place where the more technical aspects of Borel's theory are worked out.

We have now almost completed our story. Starting with the axiom of choice which Zermelo introduced in 1904 we have ended up with Brouwer's choice sequences in 1914. A note of warning—Troelstra has pointed out to us that Brouwer's use of intuitionistic logic changes the concept of choice sequence essentially. We have avoided the complications this gives the story of choice sequences by just ignoring it. For the full story, the use of the logic must also be traced.

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[2] Brouwer 1975, pp. 102–104.
[3] Borel 1972, pp. 827–878.