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# CONTEXT-SENSITIVITY AND THE TRUTH-OPERATOR IN HUGH MACCOLL'S MODAL DISTINCTIONS

MacColl finds the logical foundation for his modal distinctions in the assumption that propositions can change their truth-value depending on context. This opens the way for the introduction of a non-redundant truth-operator. Following MacColl, the expressions $A^\tau$ ("$A$ is true") and $A$ should be strictly equivalent without being synonymous. In the paper, a semantic treatment of MacColl's truth-operator is developed which fulfils these formal claims and is in accordance with the logical intuition of MacColl concerning his modal distinctions. According to this treatment, the following features of MacColl's truth-operator are stated: $A^\tau$ reports the belonging of the value *true* to $A$ in the given situation. Depending on the context of the occurrence of $A^\tau$ in an expression $B$, the value of $A$ in a given situation is transmitted to other situations which should be considered in order to evaluate $B$. By this context-fixation, the use of $A^\tau$ produces a kind of rigid designation (or rigid valuation). Consequently, in the corresponding *de-re*-treatment of the truth-operator there is a difference between being true and being certain, but there is no difference between being true and being certain of being true. This *de-re*-use of the truth-operator is accompanied by a *de-dicto*-treatment in which the strict equivalence between $A^\tau$ and $A$ holds.

## 1. Introduction

One of the main divergences of the logical intuition developed by MacColl in his *Symbolic Logic*, compared to the classical intuition of Frege consists in the fact that MacColl takes for granted that a distinction between *true* and *certain*, *false* and *impossible* is of fundamental logical relevance and that there are *variable* propositions (or statements) which are sometimes true and sometimes false. In these differences MacColl finds the basis for his modal logic of *Symbolic Logic and its Applications* (1906) with the strict implication introduced there.[1]

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MacColl finds room for his modal distinctions in orienting his logical efforts to language in use, rather than to the treatment of language as just a static system of signs. In contrast to the Fregean Aussage (proposition), a statement in MacColl's treatment is a linguistic entity, an *intelligible arrangement of words*, which is physically located in space and time and which can be psychologically perceived. One can say that this linguistic turn is fundamental to MacColl's non-classical achievements. In natural language one is immediately confronted with pragmatic characterizations like ambiguity, context-sensitivity, linguistic acts, epistemic dimensions, soundness claims, justification procedures, etc. With respect to these characterizations (but not in relation to all of them), MacColl justifies the usefulness and the logical relevance of his modal distinctions. His fundamental modal characterizations do not rest on ambiguity (one accusation of Russell's against MacColl's logical enterprise[2]). MacColl finds the main source of his non-classical departure in the context-sensitivity of language and, additionally, in different kinds of epistemic justification.

The context-sensitivity in MacColl consists not only in the presence of different applicationary or justificationary contexts, but also in a syntactic context-sensitivity of the expressions in his symbolic systems. MacColl underlines this aspect in a letter to Russell dated May 15, 1905. He writes: "This enormous superiority of my system is due in great measure to the very principle which you find so defective, namely, the principle of leaving to *context* everything in the reasoning or symbolical operations which it is not absolutely necessary to express." The application of this internal context-sensitivity principle reflects a confidence in the ability of the prospective reader to find the right connections and in his willingness to try this with charity. Nevertheless, for some devices introduced by MacColl it is difficult to catch the right connections and to come up with any sound and adequate explanation in the sense of MacColl. One very resistant place is the logical distinction between an expression $A$ and the expression predicating truth of $A$, "$A$ is true" ($A^\tau$). Already in the 1960s Storrs McCall

[1] In the course of his intellectual life, the logical views of MacColl, especially those concerning implication and modality, went through important changes. This paper does not focus on those developments, but tries to give an explication of MacColl's views concerning these topics in his main and almost final work (MacColl 1906). The intended explication should be coherent with the views uttered and the formal claims presented there. So I will not focus on the fact that MacColl held different views about modality and implication in different periods of his logical work. Even in 1906 one is confronted with the traces of views formerly favored by MacColl. For detailed information see: Astroh 1993, Rahman 1997, Rahman and Christen 1997.

[2] See Russell's review in *Mind* (1906).

closed his report about MacColl in the *Encyclopedia of Philosophy* with the words: "The many other idiosyncracies in MacColl's system, such as ... his distinction between $a$ and $a^\tau$, still await a competent interpreter" (McCall 1967, p. 546). It seems that this statement has not lost its correctness.[3] In the remainder of this paper I try to take some steps in the direction of a sound interpretation of the truth-predicate in the sense of MacColl 1906, i.e., in a sense in which MacColl would have interpreted this predicate, and which delivers more than just the indication of an intuitively possible solution, as I did in my paper of 1993 (Stelzner 1993).

MacColl finds the logical foundation for his modal distinctions in the specialty that the propositions treated in his system can change their truth-value depending on context:

> Some logicians say [Russell, e.g., did this in his review of MacColl's *Symbolic Logic and its Applications* (Russell 1906)] that it is not correct to speak of any statement as "sometimes true and sometimes false"; that if true, it must be true always; and if false, it must be false always. To this I reply ... that when I say "A is sometimes true and sometimes false", or "A is a *variable*," I merely mean that the symbol, word, or collection of words, denoted by A sometimes represents a truth and sometimes an untruth. For example, suppose the symbol A denotes the statement "Mrs. Brown is not at home." This is neither a formal certainty, like $3 > 2$, nor a formal impossibility, like $3 < 2$, so that *when we have no data but the mere arrangement of words*, "Mrs. Brown is not at home," we are justified in calling this *proposition*, that is to say, *this intelligible arrangement of words*, a variable, and in asserting $A^\theta$ ["A is a variable"]. ... To say that the proposition A is a *different proposition* when it is *false* from what it is when it is *true*, is like saying that Mrs. Brown is a *different person* when she is *in* from what she is when she is *out.* (MacColl 1906, pp. 18 f.)

According to this, we can have the same proposition in different contexts of use. The truth-value for the statements or propositions depends on the context of the application of such statements. A proposition in the Fregean-Russellian sense could be interpreted against the background of MacColl's view as a pair $\langle \text{statement}, \text{context} \rangle$. For such a pair we have a fixed truth-value, i.e. (in Frege) a fixed meaning (Bedeutung) of the proposition (Aussage), and the proposition appears to

[3] One witness to this is Shahid Rahman, who writes: "Schließlich müssen viele seiner Ideen noch kritisch erarbeitet werden, wie z.B. sein Argument für die Nicht-Synonymie der Ausdrücke '*A*' und '*A* ist wahr', die eine entscheidende Rolle in seinem Begriff der pure logic spielt." ["Eventually, many of his ideas should be critically worked out, e.g., his argument for the non-synonymy of the expressions '*A*' and '*A* is true', which plays a decisive role in his notion of pure logic."] (Rahman 1997, p. 166)

be the sense (Sinn) of the statement (Aussagesatz). Following Russell, one could conclude from this: If we give MacColl the classically right entities, namely truth-value definite propositions, to be treated in his modal system, then the modal distinctions between *true* and *certain* and between *false* and *impossible* will be superficial and MacColl's modal logic will collapse to classical logic.

Getting those classically right entities could be managed in two ways: Confining MacColl's system by the exclusion of variable statements[4] or extending the system by the introduction of symbolic tools expressing different contexts and pairs between statement and context. Neither way, I think, is in the spirit of MacColl: The general exclusion of variable statements would lead away too much from natural language use. In addition, the expression of contexts seems not to be necessary for MacColl, because he is confident that one should be able to recognize the actual context-relatedness in the context of use of a statement.

Not only I, but MacColl too would have had to agree with the above redundancy claim for both positive and negative modal distinctions if his modal distinctions rested only on context-sensitivity. But the modal distinctions in MacColl's treatment do not vanish, because, firstly, MacColl recognizes the general confinement to truth-definite propositions as unnatural and not logically forced, and, secondly, even confining our considerations to truth-value definite propositions, the modal distinctions do not vanish when we take into account MacColl's second source of this differentiation: namely, the kinds and strength of justification we have for the truth or falsehood of a (maybe truth-value definite) proposition or statement:

> A proposition is called a formal certainty when it follows necessarily from our definitions, or our understood linguistic conventions, without further data ... It is called a material certainty when it follows necessarily from some special data not necessarily contained in our definitions. (MacColl 1906, p. 97)

Contrary to the certainties mentioned here, MacColl's truth-predicate in $A^\tau$ does not depend on a kind of justification. $A^\tau$ just reports that $A$ is true:

> The symbol $A^\tau$ only asserts that $A$ is *true* in a particular case or instance. The symbol $A^\varepsilon$ asserts more than this: it asserts that $A$ is *certain*, that $A$ is always true (or true in every case) within the limits of our data and definitions, that its probability is 1. The symbol $A^\iota$ only asserts that $A$ is false in a particular

[4] MacColl considers this case, when in chapter IV he writes: "For the rest of this chapter we shall exclude the consideration of variables, so that A, $\mathrm{A}^\tau$, $\mathrm{A}^\varepsilon$ will be considered mutually equivalent, as will also $\mathrm{A}'$, $\mathrm{A}^\iota$, $\mathrm{A}^\eta$." (MacColl 1906, p. 21)

case or instance; it says nothing as to the truth or falsehood of $A$ in other instances. Thus $A^\tau$ and $A^\iota$ are simply assertive; each refers only to one case, and raises no general questions as to data or probability. (MacColl 1906, p. 7)

The explication given in the above quotation is sharpened by two formal demands which should be fulfilled in relation to $A^\tau$ and $A$: In the treatment intended by MacColl (1906), the expressions $A^\tau$ and $A$ should be strictly equivalent (this is symbolized by $A^\tau = A$), but, nevertheless, $A$ should not be synonymous with $A^\tau$ (i.e., it does not hold logically that $A^\tau$ can be replaced salva veritate in every context of its occurrence by $A$). MacColl demonstrates this non-synonymy of $A$ and $A^\tau$ by examples, but these cannot suffice as full-fledged explanation of these predicates.[5]

The use of the truth-predicate in MacColl 1906 has an extensive pre-history, beginning with MacColl's first logical writings in 1877.[6] And this pre-history shows very changing intuitions concerning the content of expressions containing the truth-predicate. Only the words "is true" remained unchanged in this history. It would be highly misleading to assume that the analysis of the early uses of this predicate could explain the use of "$^\tau$" as a symbolic form for "is true" in *Symbolic Logic* of 1906. This holds for the unsatisfactory early semantic uses of "$= 1$" as an expression for "is true" before 1906 and for the pragmatic-epistemic use of such expressions in the sense of a report about a finished decision concerning the acknowledgment or confirmation of the truth of a statement or proposition, explained by MacColl. Even if MacColl (1906) explicitly repeats such a pragmatic explanation,[7] we can take this merely as a remark about a possible use of a truth-predicate. But this pragmatic use in no way explains the semantic features of the truth-predicate "$^\tau$" used in the 1906 formalization. The pragmatic interpretation of a truth-predicate would be good for explaining the non-synonymy between "$A$" and "It is true that $A$", but in the pragmatic interpretation "$A$" and "It is true that $A$" are not equivalent to each other, and in the formal system of 1906 MacColl even claims the unrestricted soundness of the strict equivalence $A^\tau = A$.

Taking seriously the above-mentioned remark of MacColl about the

[5] I discuss these examples later.

[6] Cp. Michael Astroh's and Shahid Rahman's analysis of the changing uses of symbolic forms for the expression "is true" (Astroh 1993, Rahman and Christen 1997)

[7] "The statement $A^\tau$ is a revision and *confirmation* of the judgement A ... We suppose two incompatible alternatives, A and $A'$, to be placed before us with fresh data, and we are to decide which is true. If we pronounce in favour of A and write $A^\tau$, we *confirm* the previous judgement A and write $A^\tau$; if we pronounce in favour of $A'$, we reverse the previous judgement A and write $A^\iota$." (MacColl 1906, p. 18)

reference of $A^\tau$ to only one case, we come to the conclusion that $A^\tau$ asserts the belonging of the value *true* to $A$ in the given situation and carries this relatedness from the given context of valuation to other situations.[8] Treated this way concerning the valuation of $A$ in $A^\tau$, MacColl's predicate $^\tau$ has a context-fixing function and brings about a kind of rigid designation or rigid valuation: If the value for $A^\tau$ has to be determined in the given context in order to determine the value of an expression $H$, part of which is $A^\tau$, then the value of every subformula $B^\tau$ of $H$ is the value of $B$ in the given situation. So we should have $A$ and $A^\tau$ equivalent in every given unique imaginable situation, but we don't: If it is true that $A$ is true in the given situation (i.e., $A^\tau$), then $A$ is true in every situation. But $A^\tau$, that $A$ is true in the given situation, is then true in every other situation related to the given (formerly actual) situation: In every other situation it is true that $A$ is true in the formerly actual situation. Explicating $A^\tau$ in this way, we have a difference between being true and being certain, but we do not have any difference between being true and being certain of being true. In this sense, statements of kind $A^\tau$ would be classically right entities, i.e., entities directed to which all positive modal distinctions collapse (analogously for negative characterizations).

This informal explanation of the intuition behind the truth-predicate $^\tau$ gives a preliminary sketch of how I will try to give a sound explication of its syntax and semantics in accordance with the logical system of *Symbolic Logic*. If we attempt to give adequate semantic explications for MacColl's modal system, the problems with this intuition will be sharpened.

## 2. Syntax and Semantics for MacColl's Modal Distinctions

In order to prepare for the explication of the semantic features of the truth-predicate, I define the semantics of its non-modal and modal background in this Section.

### *2.1. The non-modal basis*

One problem with the semantics for MacColl's symbolism is the inner semantic context-sensitivity of the symbolic signs he uses. Depending on the syntactic context in which such a symbolic sign occurs, its semantic features can change.[9] This means that the semantics of

[8] Basically this is the view developed in Stelzner 1993.

[9] This context-sensitivity is characteristic for all phases of the development of MacColl's logical views, beginning with his papers in the late 1870s. Already at

MacColl's system (like semantics of natural languages) is not a purely compositionally constructed semantics. For MacColl's systems, Frege's statement that the meaning of a word has to be determined in the context of the sentence, not in isolation,[10] holds in a much stronger sense than for those of Frege himself. In this sense, MacColl makes a much more consequential use of the non-traditional dictum shared by him and Frege: "The *complete proposition* is the unit of all reasoning" (MacColl 1906, p. 11).

Depending on their occurrence in a proposition, the signs $A$, $B$, $C$, ... in MacColl's logical language can denote individual subjects or conceptual predicates:

> The symbol $\mathrm{A}^{\mathrm{B}}$ denotes a proposition of which the individual A is the subject and B the predicate. Thus, if A represents *my aunt*, and B represents *brown-haired*, then $\mathrm{A}^{\mathrm{B}}$ represents the proposition 'My *aunt* is *brown-haired*.' (MacColl 1906, p. 4)

Besides this, they can be attributes of classes and in this way they can determine subjects:

> When A is a class term, $\mathrm{A}_{\mathrm{B}}$ denotes the individual (or an individual) of whom or of which the proposition $\mathrm{A}^{\mathrm{B}}$ is true. For example, let H mean "*the horse*";

the early stage of his work, MacColl sees it not as a defect, but as a tool which he uses in order to avoid introducing new symbolic signs and unfamiliar-looking formal explanations: "Strange-looking symbols somehow offend the eye; and we do not take to them kindly, even when they are of simple and easy formation. Provided we can avoid ambiguity, it is generally better to intrust an old symbol with new duties than to employ the services of a perfect stranger. In the case just considered, and in many analogous cases, the context will be quite sufficient to prevent us from confounding one meaning with another, just as in ordinary discourse we run no risk of confounding the meanings of the word *air* in the two statements—'He assumed an air of authority,' and 'He resolved the air into its component gases.'" (MacColl 1882, pp. 229 f.). The context determines not just the meaning of descriptive signs, like "air" in the given example. The syntactic context in some cases determines even to what category a special sign belongs. In most cases the context says enough for a clearly unambiguous reconstruction. But in other cases the flavor of ambiguity cannot be avoided. The danger of ambiguity is increased by the intensive development of MacColl's logical views. So I agree with Michael Astroh, when he (after the examination of MacColl's use of $A = 1$) comes to the conclusion: "Aufgrund der Korrekturen, die MacColl von einem Aufsatz zum nächsten einfließen läßt, ohne die Konsequenzen seiner Interpretationen zu diskutieren, ist es zumindest im Hinblick auf seine frühen Schriften schwierig, ihm von vornherein eine einheitliche Position zu unterstellen, die er nur zunehmend genauer zu artikulieren wüßte." ["Because of the corrections which MacColl introduces from one paper to the next without discussing the consequences of his interpretations, it is at least in respect of his early writings difficult to presuppose for MacColl a homogeneous position which he is able to articulate more and more precisely."] (Astroh 1993, p. 132).

[10] "Nach der Bedeutung der Wörter muß im Satzzusammenhange, nicht in der Vereinzelung gefragt werden." (Frege 1884, p. X)

let w mean "*it won the race*"; and let s mean "*I sold it*," or "*it has been sold by me*." Then $\mathrm{H}_{\mathrm{w}}^{\mathrm{s}}$, which is short for $(\mathrm{H}_{\mathrm{w}})^{\mathrm{s}}$, represents the complex proposition "The *horse* which *won* the race has been *sold* by me," or "I have sold the horse which won the race" ... Thus the suffix w is *adjectival*; the exponent s *predicative* ... The symbol $\mathrm{H}^{\mathrm{w}}$, without an adjectival suffix, merely asserts that a horse or the horse, won the race without specifying which horse of the series $\mathrm{H}_1, \mathrm{H}_2$, &c. (MacColl 1906, p. 4 f.)

In the propositional logic (MacColl speaks in this case about *pure* or *abstract* logic) the same symbols $A, B, C, \ldots$ can denote statements or propositions (cf. MacColl 1906, p. 6). So, in defining the (non-modal) syntax of the system, we have just one kind of non-logical terms, predicate-/class-/subject-/individual-/statement-/proposition-terms: $A, B, C, \ldots$ (short: P-terms).

MacColl introduces in his system a denial (or negation) which is directed (again like the Frege-negation) to sentences or propositions, but does not constitute negative terms (even if the syntactic place of its occurrence could suggest so):

A small *minus* before the predicate or exponent, or an acute accent affecting the whole statement, indicates denial. (MacColl 1906, p. 5)

For the expression of sentence negation MacColl uses an accent as well as the small minus. $(\alpha^{\beta})'$ expresses the same as $\alpha^{-\beta}$: As mentioned above, he does not introduce special negations for predicate terms; affirmation and negation are the only qualities for sentences in MacColl: He does not have the Kantian infinite judgements, i.e., judgements with negative predicates, as a special kind of affirmative judgements. As two-placed propositional connectives he introduces the classical disjunction $(A^C + B^D)$ and conjunction $(A^C B^D)$.

Based on this material one can compose different kinds of expressions for statements:

If $\alpha, \beta, \gamma$ are P-terms, then single occurrences of $\alpha, \alpha', \alpha^{*\beta}, \alpha_{*\beta}^{*\gamma}$ are statements, where $'$ expresses the denial of a statement, $*$ indicates the place where the small minus can be or not. If $S_1$ and $S_2$ are statements, then $S_1', S_1 S_2$ and $S_1 + S_2$ are statements.

The syntax introduced this way is loaded with semantic presupposition. Every use of $\alpha_{*\beta}$ in a statement presupposes $\alpha^{*\beta}$:

The symbol $\mathrm{H}_{\mathrm{C}}$ ("The *caught horse*") *assumes* the statement $\mathrm{H}^{\mathrm{C}}$, which *asserts* that "The *horse* has been *caught*." Similarly $\mathrm{H}_{-\mathrm{C}}$ assumes the statement $\mathrm{H}^{-\mathrm{C}}$. (MacColl 1906, p. 5)

The fact that we are confronted here with presupposition is shown by the fact that $H^C$ is assumed both if the whole sentence in which it occurs is affirmative $(H_C^A)$ and if it is negative ($H_C^{-A}$ or $(H_C^A)'$).

As a special class MacColl introduces the class of non-existing things:

The symbol 0 denotes *non-existence*, so that $0_1, 0_2, 0_3$, &c., denote a series of names or symbols which correspond to nothing in our universe of admitted realities. (MacColl 1906, p. 5)

This zero-class plays an important role for expressing quantified categorical propositions. According to MacColl, we can express the categorical judgments of traditional logic in the following way:

| | |
|---|---|
| $\alpha^0_\beta$ | "No $\alpha$ is $\beta$" |
| $\alpha^{-0}_\beta$ | "Some $\alpha$ are $\beta$" |
| $\alpha^0_{-\beta}$ | "Every $\alpha$ is $\beta$" |
| $\alpha^{-0}_{-\beta}$ | "Some $\alpha$ are not $\beta$" |

The existence presupposition mentioned above, according to which every use of $\alpha_{*\beta}$ in a statement presupposes $\alpha^{*\beta}$, clearly does not hold in the case of categorical judgements, because then "No $\alpha$ is $\beta$" ($\alpha^0_\beta$) would assume "$\alpha$ is $\beta$" ($\alpha^\beta$). This in some sense contradicts the overall stipulation of MacColl's that every use of $H_C$ assumes $H^C$, and we should correct it in the following way: Every use of $\alpha_{*\beta}$ in a statement, the predicate of which does not equal 0, presupposes $\alpha^{*\beta}$. I mention this here because it is another example of MacColl's context-sensitive use of his symbolic language: Sometimes even simple substitutions have to be structurally treated in a way other than the expression in which the substitution was performed. I shall keep this in mind in undertaking to explicate the syntactic use and semantic role of the truth-predicate $^\tau$.

### *2.2. The modal distinctions in pure or abstract logic*

For his propositional logic, MacColl uses the expression *pure* or *abstract* logic. He characterizes his modal distinctions as follows:

In *pure* or *abstract* logic statements are represented by single letters, and we classify them according to attributes as *true*, *false*, *certain*, *impossible*, *variable*, respectively denoted by the five Greek letters $\tau, \iota, \varepsilon, \eta, \theta$. Thus the symbol $A^\tau B^\iota C^\varepsilon D^\eta E^\theta$ asserts that A is *true*, that B is *false*, that C is *certain*, that D is *impossible*, that E is *variable* (possible but uncertain). The symbol $A^\tau$ only asserts that A is true in a particular case or instance. The symbol $A^\varepsilon$ asserts more than this: it asserts that A is certain, that A is always true (or true in every case) within the limits of our data and definitions, that its probability is 1. The symbol $A^\iota$ only asserts that A is false in a particular case or instance; it says nothing about the truth or falsehood of A in other instances. The symbol $A^\eta$ asserts more than this; it asserts that A contradicts some datum or definition, that its probability is 0. Thus $A^\tau$ and $A^\iota$ are simply *assertive*; each refers to only one case, and raises no general questions as to data or probability. The symbol $\mathrm{A}^{\theta}$ (A is a *variable*) is equivalent to $\mathrm{A}^{-\eta}\mathrm{A}^{-\varepsilon}$; it asserts that A is neither *impossible* nor *certain*, that is, that A is *possible* but *uncertain*. (MacColl 1906, pp. 6 f.)

Based on the given modal characterizations, MacColl defines the strict implication between $A$ and $B$ as the impossibility of the conjunction between $A$ and non $B$:

The symbol $\mathrm{A}^{\mathrm{B}} : \mathrm{C}^{\mathrm{D}}$ is called an *implication*, and means $(\mathrm{A}^{\mathrm{B}}\mathrm{C}^{-\mathrm{D}})^{\eta}$, or its synonym $(\mathrm{A}^{-\mathrm{B}} + \mathrm{C}^{\mathrm{D}})^{\varepsilon}$. It may be read in various ways, as (1) $\mathrm{A}^{\mathrm{B}}$ implies $\mathrm{C}^{\mathrm{D}}$; (2) If A belongs to the class B, then C belongs to the class D; (3) It is impossible that A could belong to the class B without C belonging to the class D. (MacColl 1906, p. 7)

MacColl gives a sequence of "self-evident or easily proved formulae": these are of special importance for the explication of his semantic treatment of modalities (1906, p. 8; MacColl's numbering).[11]

$$(11)\ (A + A')^{\varepsilon};\ (12)\ (A^{\tau} + A^{\iota})^{\varepsilon};\ (13)\ (AA')^{\eta};\ (15)\ A^{\varepsilon} : A^{\tau};$$
$$(16)\ A^{\eta} : A^{\iota};\ (17)\ A^{\varepsilon} = (A')^{\eta};\ (18)\ A^{\eta} = (A')^{\varepsilon};\ (19)\ A^{\theta} = (A')^{\theta};$$
$$(20)\ \varepsilon : A = A^{\varepsilon};\ (21)\ A : \eta = A^{\eta};\ (22)\ A\varepsilon = A;\ (23)\ A\eta = \eta.$$

With some of these formulae, we are again confronted by the syntactic context sensitivity of MacColl's logical language. In (20)–(23) the modal signs $\varepsilon$ and $\eta$ occur in different syntactic and semantic functions: as symbols for statements that are certain and as symbols for the predicate "is certain" (analogously for "impossible"). However, we have to acknowledge that the context of MacColl's use is clear enough, so that we can, depending on the position of these signs, sharply discern these different functions. In this case, there is no reason to accuse MacColl by saying that his syntactic uniform use would lead to ambiguity.

*2.3. Classical non-classical semantics for pure logic*

At the beginning, we mentioned that, following MacColl, a statement could sometimes be true and sometimes false. It would be misleading to conclude from this that MacColl was a supporter or even a forerunner of paraconsistent logic. At the same time (or in the same situation), there is no possibility for the same statement to be true and false. A statement's being true and false is an impossibility: As sound formulae in MacColl we have $(AA')'$, $(AA')^{\eta}$ and $(A^{\tau}A^{\iota})^{\eta}$.

[11] As an abbreviation for $(A : B)(B : A)$ MacColl introduces $A = B$.

One remark of MacColl's in *Symbolic Logic and its Applications* gives the impression that he acknowledged that a statement $A$ can be neither true nor false. He even introduces a symbol in order to express this: "$\phi^0$ asserts that $\phi$ is a *meaningless* statement which is neither true nor false" (MacColl 1906, p. 10). In fact, such meaningless statements are easy to produce in the framework of MacColl: Because the use of $H_C$ assumes $H^C$, any statement $S$ containing $H_C$ is meaningless if $H^C$ is not true. In this case $S$ is neither true nor false. However, no consequences follow from this for the logical system in the sense that there would be truth-value gaps or three values *true*, *false* and *meaningless*. Logic has to do only with admissible values and *no meaning* is not an admissible value for a statement. This follows from the statement preceding the above quotation:

> The symbol $\phi^\varepsilon$ asserts that $\phi$ is *certain*, that is, true for all admissible values (or meanings) of its constituents; the symbol $\phi^\eta$ asserts that $\phi$ is *impossible*, that is, false for all admissible values (or meanings) of its constituents; the symbol $\phi^\theta$ means $\phi^{-\varepsilon}\phi^{-\eta}$, which asserts that $\phi$ is *neither certain nor impossible*. (MacColl 1906, p. 10)

Furthermore, MacColl takes $A + A'$, $(A + A')^\varepsilon$ and $(A^\tau + A^\iota)^\varepsilon$ to be sound formulae. According to this, MacColl admits in his logic only the values true and false for statements and propositions. We here have a close similarity between MacColl 1906 and Frege's *Begriffsschrift* from 1879. As arguments of his content-stroke (Inhaltsstrich) Frege admits only judgeable contents (beurteilbare Inhalte), i.e., true or false propositions. This picture changes with the Frege of the *Grundgesetze der Arithmetik*. Here, after the introduction of the horizontal-stroke (Waagerechter), all values are admissible.

MacColl explicates his intuitive semantic principles by examples and by pointing to formulae which should be sound in his treatment. I will try to systematize MacColl's semantic intuition and to build up a formal semantics for his pure logic. In this semantics, we shall have means to express the context-sensitivity of valuations and we will not speak just about the value of an expression but about the value of an expression in a context (or in a situation).

We will use as abbreviations, with $k$ as an element of a set of possible contexts (situations, worlds, cases):

For "The value of $A$ in $k$ is *True*": $v_k(A) = T$.
For "The value of $A$ in $k$ is *False*": $v_k(A) = F$.

*2.3.1. The classical non-modal principles:*

P1. Either $v_k(A) = T$ or $v_k(A) = F$

P2. $v_k(A') = T \Longleftrightarrow v_k(A) = F$

P3. $v_k(AB) = T \Longleftrightarrow v_k(A) = T$ and $v_k(B) = T$

P4. $v_k(A + B) = F \Longleftrightarrow v_k(A) = F$ or $v_k(B) = F$

Here P1 states that in the sense of MacColl's we have only $T$ and $F$ as admissible values for statements.

Definitions:

D1. $A \supset B =_{\text{df}} (AB')'$

D2. $A \leftrightarrow B =_{\text{df}} (A \supset B)(B \supset A)$

*2.3.2. The classical modal principles:*

Since for MacColl certainty is intuitively explained just as truth in all possible situations, in the following I will not use relational semantics for the modal part. Other developments of MacColl's system would of course involve the introduction of relational semantics. I start with some clear and non-controversial principles explained by MacColl.[12]

PC. $v_k(A^\varepsilon) = T \Longleftrightarrow \forall k(v_k(A) = T)$

PI. $v_k(A^\eta) = T \Longleftrightarrow \forall k(v_k(A) = F)$

PV. $v_k(A^\theta) = T \Longleftrightarrow \neg\forall k(v_k(A) = T)$ and $\neg\forall k(v_k(A) = F)$

Definitions:

D3. $A : B =_{\text{df}} (A \supset B)^\varepsilon$

D4. $A = B =_{\text{df}} (A : B)(B : A)$

Soundness:

> A formula $A$ is sound ("$\models A$") if and only if for every set of contexts in every possible valuation $k$: $v_k(A) = T$.

Because such distinctions play an essential role in MacColl's argument for the non-redundancy of the truth predicate, I will now formulate semantic rules for the modal distinctions in the position of statements or propositions. As we saw above, MacColl sometimes uses $\varepsilon$, $\eta$ and $\theta$ in the position of statements. Thus, we find expressions like

[12] In the following, the signs $\neg$, $\forall$ and $\exists$ are used in the metalanguage.

$\varepsilon^{\varepsilon}$, $\eta^{\varepsilon}$ and $\theta^{\varepsilon}$ or $\varepsilon^{\eta}$, $\eta^{\tau}$ and $\theta^{\iota}$, etc. It would be misleading to treat the statement-modalities as logical constants, standing for "The Necessity", "The Impossibility", "The Variable" or as special truth-values "Necessary", "Impossible", "Variable". These expressions should be interpreted as variables for statements or propositions of different sorts, namely $\varepsilon$ as a variable for statements which are certain, $\eta$ as a variable for impossible statements and $\theta$ as a variable for variable statements. Subscripts on these sort-variables indicate that we have different variables of the same sort. According to this we have:

PC*. $\quad v_k(\varepsilon_i) = v_k(A_i)$, with $\forall m(v_m(A_i) = T)$

PI*. $\quad v_k(\eta_i) = v_k(B_i)$, with $\forall m(v_m(B_i) = F)$ and

PV*. $\quad v_k(\theta_i) = v_k(C_i)$, with $\exists m(v_m(C_i)) = T$ & $\exists m(v_m(C_i)) = F$

We have, e.g., $\models \varepsilon_i : \varepsilon_j$, $\models \varepsilon_i = \varepsilon_j$, $\models \eta_i : \eta_j$, $\models \eta_i = \eta_j$, $\models \theta_i : \theta_i$, or shorter $\models \theta : \theta$, but we don't have $\theta_i : \theta_j$ as a sound expression, because $v_k(C_i) = v_k(C_j)$ does not hold in every valuation $k$ for different variable statements $C_i$ and $C_j$. The logical difference between on the one hand $\varepsilon, \eta, \theta$ taken as variables for propositions, which are sorted according to their modal characterizations, and on the other hand the use of these symbols as modal operators in $A^{\varepsilon}$, $A^{\eta}$ and $A^{\theta}$, is underlined by the following: While we have as sound $\models A^{\theta} = (A^{\varepsilon})'(A^{\eta})'$, the expression $\theta = \varepsilon'\eta'$ is not sound. While $\theta$ stands for a proposition sometimes true and sometimes false, $\varepsilon'\eta'$ expresses a plain contradiction, which is never true.

There are sound expressions often referred to as "paradoxes of strict implication". Based on his classical attitude, MacColl argues convincingly for the soundness of such paradoxes:

> Symbolic logic too has its paradoxes, that is to say, formulae which appear paradoxical till they are explained, and then cease to be paradoxes. Such is the formula $\eta : \varepsilon$, which asserts that "an impossibility implies a certainty". As soon as we define the implication A : B, by which we symbolize the statement that "A implies B," to mean simply $(\mathrm{AB}')^{\eta}$, which asserts that the affirmation A coupled with the denial B′ contradicts our data or definitions, the paradox vanishes. For then $\eta : \varepsilon$ is seen simply to mean $(\eta\varepsilon')^{\eta}$, which is a clear truism. (MacColl 1906, p. 505)

Again, in accordance with our explication of the treatment of modal terms in statement-position as variables of special sorts, the sound expressions $\models \eta : \varepsilon$ and $\models (\eta\varepsilon')^{\eta}$ as well as $\models \theta : \varepsilon$, $\models \eta : \theta$ should not be confused with the respectively unsound ones $A^{\eta} : A^{\varepsilon}$, $(A^{\eta}A^{-\varepsilon})^{\eta}$, $A^{\theta} : A^{\varepsilon}$, $A^{\eta} : A^{\theta}$.

## 3. Semantics for the Truth-Predicate

One unsolved problem in explaining the semantic background of MacColl's system consists in the explication of the semantic features of the expressions $A^\tau$ ($A$ is true) and $A^\iota$ ($A$ is false), according to the intuition that $A^\tau$ should be equivalent but not synonymous with $A$ (not substitutable with $A$ in every occurrence):

It may seem paradoxical to say that the proposition A is not quite synonymous with $\mathrm{A}^\tau$, nor $\mathrm{A}'$ with $\mathrm{A}^\iota$ : yet such is the fact. Let A=*It rains.* Then $\mathrm{A}' =$ *It does not rain*; $\mathrm{A}^\tau =$ *It is true that it rains;* and $\mathrm{A}^\iota =$ *It is false that it rains.* The two propositions A and $\mathrm{A}^\tau$ are *equivalent* in the sense that each *implies* the other; but they are not *synonymous*, for we cannot always substitute the one for the other. In other words, the equivalence $(\mathrm{A} = \mathrm{A}^\tau)$ does not necessarily imply the equivalence $\phi(\mathrm{A}) = \phi(\mathrm{A}^\tau)$. For example let $\phi(\mathrm{A})$ denote $\mathrm{A}^\varepsilon$; then $\phi(\mathrm{A}^\tau)$ denotes $(\mathrm{A}^\tau)^\varepsilon$ ... Suppose now that A denotes $\theta_\tau$, a variable that turns out true, or happens to be true in the case considered, though it is not true in all cases. We get

$$\phi(\mathrm{A}) = \mathrm{A}^\varepsilon = \theta_\tau^\varepsilon = (\theta_\tau)^\varepsilon = \eta;$$

for a variable is never a certainty, though it may turn out true in a particular case.

Again, we get

$$\phi(\mathrm{A}^\tau) = (\mathrm{A}^\tau)^\varepsilon = (\theta_\tau^\tau)^\varepsilon = \varepsilon^\varepsilon = \varepsilon$$

... In this case, therefore, though we have $\mathrm{A} = \mathrm{A}^\tau$, yet $\phi(\mathrm{A})$ is not equivalent to $\phi(\mathrm{A}^\tau)$. (MacColl 1906, p. 16)[13]

MacColl believes these distinctions between $A$ and "$A$ is true" to be of fundamental cultural importance:

It is a remarkable fact that nearly all civilized languages, in the course of their evolution, as if impelled by some unconscious instinct, have drawn this distinction between a simple affirmation A and the statement $\mathrm{A}^\tau$, that A is *true*; and also between a simple denial $\mathrm{A}'$ and the statement $\mathrm{A}^\iota$, that A is *false*. (MacColl 1906, pp. 17 and 513 f.)

Nevertheless, one gets into some trouble if one tries to give a fitting explication for this distinction in the framework for a formal semantics sketched above, because here we have a case where a system seems to be neither extensional nor intensional in the sense of Carnap, yet seems to work logically properly.

[13] An analogous result was developed by MacColl in relation to $\mathrm{A}'$ and $\mathrm{A}^\iota$. In the following we discuss the problem with $\mathrm{A}'$ and $\mathrm{A}^\iota$ not separated from A and $\mathrm{A}^\tau$, but consider $\mathrm{A}^\iota$ just as $(\mathrm{A}')^\tau$.

Following the line of argumentation from the introduction, I will discuss three promising attempts for such an explication: First, I will treat $^\tau$ as an actuality-operator which gives strict equivalence between $A$ and $A^\tau$. Second, stressing the cross-reference to the case considered, the non-synonymy between $A$ and $A^\tau$ comes out. And third, semantic means will be developed which allow the first and the second treatments to be brought together. This results in the fulfilment of MacColl's demand, according to which we should have both strict equivalence and non-synonymy of $A$ and $A^\tau$.

*3.1. A first attempt: Fixing the actual*

One possible way to get fitting semantic stipulations for expressions with the truth-predicate could be expected by confining the reference of expressions in the scope of the truth predicate $^\tau$ (like $A$ in $A^\tau$) to a fixed actual context $c_a$. M. Davies and L. Humberstone give a similar treatment (not directed to MacColl) for the actuality operator:

> We need to extend our modal language by the addition of an operator '$A$' corresponding to the adverb 'actually'. The main semantic feature of such an operator is that for any sentence $\sigma$, $\lceil A\sigma \rceil$ is true with respect to a given possible world just in case $\sigma$ is true with respect to the actual world (that is, just in case $\sigma$ is actually true). (Davies and Humberstone 1980, p. 221)[14]

In accordance with the core idea of 'actuality', we can give a first semantic explication of MacColl's truth predicate $\tau$:

PT1. $$v_k(A^\tau) = T \Longleftrightarrow v_{c_a}(A) = T$$

However, this confinement to one distinguished actual situation, which is independent of the given context of valuation, seems to be contrary to the pragmatic normal language orientation of MacColl's. There should be different possible situations which can give different actual contexts (or cases considered), not only one context-fixing eternal actual context in which we can use "$A$ is true". Clearly, MacColl's formal claims are not fulfilled in a treatment of the predicate $^\tau$ according to PT1: his intention to have $A$ and $A^\tau$ be equivalent expressions in any case considered is not fulfilled, i.e., instead of the sound $A = A^\tau$, we have only $(A = A^\tau)^\tau$.

In order to overcome this deviation concerning the strict equivalence between $A$ and $A^\tau$, one could revise PT1 in the sense that the idea of a fixed actual context is given up and the context of valuation taken as the actual context, which changes according to different valuations. This would lead to the semantical principle $v_k(A^\tau) = T \Longleftrightarrow c_a = k$ and $v_{c_a}(A) = T$. This can be simplified to PT2:

[14] Other relevant papers are Crossley and Humberstone 1977 and Davies 1981.

PT2. $$v_k(A^\tau) = T \Longleftrightarrow v_k(A) = T$$

Then we have

$$\text{For all } k : v_k(A^\tau) = v_k(A).$$

This seems to give what MacColl asserts: $A^\tau$ and $A$ are materially and strictly equivalent. We don't relate $A^\tau$ to a special designated actual world, but to "the case considered", as MacColl demands. According to PT2, $\models_{PT2} A^\tau \leftrightarrow A$, and $\models_{PT2} A^\tau = A$ are sound expressions. However, what about the asserted non-synonymy of the statements $A^\tau$ and $A$? The given condition PT2 considers the operator $^\tau$ in $A^\tau$ as semantically redundant. Thus, for every context $k$ we have the same value for $A^\tau$ and $A$. From this we can replace $A^\tau$ by $A$ everywhere without changing the truth-value of the imbedding sentence. The expressions $A^\tau$ and $A$ are then synonymous, contrary to MacColl's intuition.

*3.2. Reference to the case considered*

Evidently, with PT2 we did not succeed in giving an adequate explication of a semantics behind MacColl's intuition and his formal claims for the use of the expression "$A$ is true" ($A^\tau$). We caught only the unsurprising part, where $A^\tau$ is equivalent with $A$: In every context, $A$ and $A^\tau$ are both true or both false. Accordingly, our task now should be to catch the surprising part, where $A$ is not synonymous with $A^\tau$, but $A$ is nevertheless equivalent to $A^\tau$. The breakdown of synonymy between $A$ and $A^\tau$ is connected with possible occurrences of $A$ and $A^\tau$ inside the scope of modal operators. Here the context-fixing function of $^\tau$ develops its logical significance: Similar to a rigid designation, with $^\tau$ we can produce a kind of *de-re*-valuation inside modal contexts. Then inside the scope of modal operators, there can be a difference between the values of $A$ not connected with $^\tau$ (which has to be treated *de dicto*) and the values of $A$ in $A^\tau$ (which has to be treated *de re*, taking its values from the valuation outside the scope of the modal operator).

In order to prepare for the semantic explication of the context-fixing role of the truth-predicate $^\tau$ we introduce a new syntactic device for the expression of a special context-fixing predicate, which refers to specific explicitly indicated contexts.[15] $A^i$ means that "$A$ is true in context $i$", according to the semantic rule:

PT$^{\text{f}}$. $$v_k(A^i) = T \Longleftrightarrow v_i(A) = T$$

[15] In the rest of this paper the symbols $i$, $j$, $k$ etc. are used in the metalanguage as symbols for contexts; they are used in the object language as predicates governed by the semantic rule PT$^{\text{f}}$.

Introduced in this way $A^i$ delivers a kind of context-invariant proposition, which we do not have in MacColl. We will use this context-fixing operator to determine $A^\tau$, which is in a special sense context-invariant.

The actuality-predicate, as treated in PT1, is just one special case of the truth-predicates introduced with $\mathrm{PT}^{\mathrm{f}}$.

The context-independence of $A^i$ given by $\mathrm{PT}^{\mathrm{f}}$ leads to the following:

$$(*) \qquad v_k(A^i) = T \Longleftrightarrow v_m(A^i) = T$$

As mentioned in connection with the Russell–MacColl discussion and concerning the case where we do not have variable statements,[16] for such context-invariant statements the modal distinctions between being true and being certain (or necessary) have no logical significance:

$$v_k(A^i) = T \Longleftrightarrow v_m(A^{i\varepsilon}) = T.$$
$$v_k(A^{-i}) = T \Longleftrightarrow v_m(A^{i\eta}) = T.$$

Or, syntactically:

$$\models A^i = A^{i\varepsilon}.$$
$$\models A^{-i} = A^{i\eta}.$$

Nevertheless, we don't have: $\models A^\varepsilon = A$ (or $\models A' = A^\eta$).

So, $A^i$ and $A$ are not synonymous. However, they are not equivalent either, because we do not have

$$v_k(A^i) = T \Longleftrightarrow v_k(A) = T.$$

Accordingly, we have neither $\models A^i = A$ nor $\models A^i \leftrightarrow A$.[17] However, with the truth-predicates of the kind just introduced we did not connect a claim that MacColl's truth-predicate $^\tau$ would be one of those truth-predicates. Truth-predicates $^i$, $^j$, $^k$, etc. will play a supporting role in the explication of the truth-predicate $^\tau$. They will allow us to combine the merits of PT1 and PT2 for the semantic explication of MacColl's $^\tau$ in a technically simple way: In a valuation of a formula $H$ in a given context $k$ the expressions $A$ and $A^\tau$ should be equally evaluated in non-modalized contexts, but in modalized contexts the valuation of $A^\tau$ should be based on a fixed *de-re*-reference to the context $k$ in which the

[16] See page 94.

[17] Of course (cf. the actuality-treatment with PT1), we have as sound formulae:

$$\models (A^i \leftrightarrow A)^i.$$
$$\models (A^i = A)^i.$$

whole formula $H$ is evaluated (the "actual" world, which can change with every valuation of $H$). This leads to the following semantical rule: In evaluating a formula $H$, we fix all values of subformulae $A$ which are governed by the truth predicate $^\tau$ to the value of $A$ in the valuation considered for the formula $H$.

PT3. $$v_k(H) = T \Longleftrightarrow v_k(H^*) = T,$$

where we get $H^*$ from $H$ by replacing all occurrences of $^\tau$ with $^k$.

In this treatment, the truth-predicate $^\tau$ is formally equivalent to the predicate "now" treated by Hans Kamp (1971).[18] The main difference concerns the framework of explication: While Kamp uses a device of double indication of the truth-value of a formula ("$H$ is true relative to situation $s_1$ and relative to situation $s_2$"), we avoid the double indication of context-relatedness by the use of the context-fixing predicates $^i$, $^j$, $^k$, etc. One can wonder whether the double indexing framework or our substitution framework is better suited for different tasks. We concentrate our efforts on the special task of giving an explication of MacColl's truth-predicate $^\tau$, which should be close to the intuitions and formal claims of MacColl's concerning this predicate. So, we shall examine the results of the level of explication now reached for expressions $A^\tau$, which in fact (having in mind remarks of MacColl's tending to a time-logic interpretation of his modalities) could be read not only as "$A$ is true in the case considered", but also as "$A$ is true *now*".

Given PT1, the following semantic principle prevented the equivalence between $A$ and $A^\tau$:

$$v_k(A^\tau) = T \Longleftrightarrow v_m(A^\tau) = T.$$

This does not follow from PT3; however, we have:

$$v_k(A^\tau) = T \Longleftrightarrow v_k(A) = T, \text{ for all possible contexts } k.$$

Therefore, we have as sound the equivalence between $A$ and $A^\tau$,

(1) $$\models_{PT3} A^\tau \leftrightarrow A,$$

i.e., $A^\tau$ and $A$ are materially equivalent in every valuation. And it holds, as claimed by MacColl, that $A^\tau$ and $A$ are not synonymous. The modal characterizations of being true and of being necessary that it is true are strictly equivalent:

(2) $$\models_{PT3} A^\tau = A^{\tau\varepsilon}.$$

[18] Prior 1968, Burgess 1984, Fenstad et al. 1987, van Benthem 1988, 1991, Gundersen 1997.

From this, together with the unsoundness of

(3) $$\models_{PT3} A = A^{\varepsilon},$$

we obtain (in accordance with MacColl) that $A$ and $A^{\tau}$ are not synonymous. But they are synonymous in their unmodalized occurrences. So, from the soundness of (1) and (2) we obtain the soundness of

(4) $$\models_{PT3} A = A^{\tau\varepsilon}.$$

*3.3. The case considered and the Gödel-rule*

The results mentioned so far witness to the appropriateness of our explication of the truth-predicate $^{\tau}$ according to PT3. But there is one serious problem unsolved: MacColl claims not only the soundness of the material equivalence between $A$ and $A^{\tau}$, but also the soundness of the strict equivalence between them. And this does not hold with PT3.

We do not have—contrary to the claim of MacColl's—that $A^{\tau}$ and $A$ are strictly equivalent with each other, i.e.,

(5) $$\models_{PT3} A^{\tau} = A$$

does not hold. The soundness of $\models_{PT3} A^{\tau} \leftrightarrow A$ together with the unsoundness of (5) demonstrates that with $\mathrm{PT^f}$ and PT3 the Gödel-Rule

$$\text{if } \models_{PT3} H, \text{ then } \models_{PT3} H^{\varepsilon}$$

does not hold.

One way to overcome the trouble with the Gödel-Rule (and in this way to ensure the soundness of $\models A^{\tau} = A$, as demanded by MacColl) consists in differentiating expressions $H$ in which the truth predicate $^{\tau}$ occurs *unmodalized* (outside every subformula of kind $G^{\varepsilon}$, $G^{\eta}$, $G^{\theta}$, $G_1 : G_2$ and $G_1 = G_2$) from expressions where all occurrences of the truth predicate $^{\tau}$ occur *modalized* (located inside modal contexts). Principle PT3 will be replaced by the following principle PT4:

PT4. If there are unmodalized occurrences of $^{\tau}$ in a formula $H$, then, before applying other semantic rules, the valuation of $H$ has to proceed according to the following scheme:

$$v_k(H) = T \Longleftrightarrow v_k(H^*) = T,$$

where we obtain $H^*$ from $H$ by replacing all (modalized and unmodalized) occurrences of $^{\tau}$ with $^{k}$.
(Expressions of kind $A^{\iota}$ are treated as $(A')^{\tau}$.)

The difference between PT3 and PT4 lies in the condition for the replacement of $^{\tau}$ by the truth-predicate $^{k}$ corresponding to the given valuation $k$. With PT4 the substitution of $^{k}$ for all (modalized and unmodalized) occurrences of $^{\tau}$ only needs to be performed in formulae in which $^{\tau}$ occurs unmodalized. So, with PT3 we have a general *de-re*-treatment of $^{\tau}$, while with PT4 we distinguish the *de-re*-treatment of $^{\tau}$ in formulae with unmodalized occurrences of $^{\tau}$ from the *de-dicto*-treatment of $^{\tau}$ in formulae without unmodalized occurrences of $^{\tau}$. This leads to the result that formulae of type $H^{\tau}$ are sound in the treatment with PT3 if and only if $H^{\tau}$ is sound in the treatment with PT4:

(I) $$\models_{PT3} H^{\tau} \iff \models_{PT4} H^{\tau}.$$

With

(II) $$\models_{PT3} H^{\tau} \iff \models_{PT3} H$$

this gives

(III) $$\models_{PT3} H \iff \models_{PT4} H^{\tau}.$$

Characteristic differences between the PT3- and the PT4-treatment can be exemplified by the following comparison: While in the PT3-treatment we have both (1) $H \supset H^{\tau\varepsilon}$ and (2) $H^{\tau} \supset H^{\tau\varepsilon}$ sound, only (2) is in the PT4-treatment. However, in accordance with (II), from $\models_{PT3} H \supset H^{\tau\varepsilon}$ we get soundness of $\models_{PT4} (H \supset H^{\tau\varepsilon})^{\tau}$.

The specific possibility to express a *de-dicto*-treatment of $^{\tau}$ in the PT4-treatment validates the Gödel rule,

(IV) $$\models_{PT4} H \implies \models_{PT4} H^{\varepsilon},$$

which does not hold with the PT3-treatment. There are formulae sound in PT4 (like $A = A^{\tau}$) which are not sound in PT3. This is the case because of the different treatments of $^{\tau}$ in $A = A^{\tau}$: With PT3 it is treated *de-re*, with PT4 *de-dicto*.

Because of the difference between the *de-re*-treatment and the *de-dicto*-treatment given with PT4, even given soundness of the strict equivalence between $A$ and $A^{\tau}$, there is no equivalence concerning the soundness of separate formulae $A$ and $A^{\tau}$: It is not the case that $\models_{PT4} H \iff \models_{PT4} H^{\tau}$. E.g., in accordance with (III) and the unsoundness of $H = H^{\tau}$ with PT3, $(H = H^{\tau})^{\tau}$ is not sound with PT4, but $\models_{PT4} H = H^{\tau}$ is.

With the rule PT4 (as formerly with PT3) we connected the valuation for expressions of kind $A^{\tau}$ with the syntactic structure of such expressions, but also took into consideration the valuation context (or the actual context of use) for this expression. In some sense we have managed to bring together compositionality and context: If a formula $H$ contains unmodalized occurrences of $^\tau$, then in a first step all such formulae are bound to the given evaluation context. Further calculations are carried out not with expressions of kind $A^\tau$, which have been evaluated in a subformula according to special valuation contexts of this subformula, but with expressions of kind $A^k$, which are context-insensitive, as shown in the semantic relation $v_k(A^i) = T \Longleftrightarrow v_m(A^i) = T$ and in the corresponding sound formula $\models A^{ik} = A^{im}$.

With respect to the explication of MacColl's views, the treatment of $\tau$ according to PT4 brings about the following results:

STRICT EQUIVALENCE OF $A^\tau$ AND $A$. $\models A^\tau = A$.

*Proof.* For every $m$:

| | | |
|---|---|---|
| (1) | $v_m(A^\tau = A) = T \Longleftrightarrow \forall k(v_k(A^\tau \leftrightarrow A) = T)$ | D2, D3, D4, PC |
| (2) | $v_k(A^\tau \leftrightarrow A) = T \Longleftrightarrow v_k(A^k \leftrightarrow A) = T$ | PT4 |
| (3) | $v_k(A^k \leftrightarrow A) = T \Longleftrightarrow v_k(A^k) = v_k(A)$ | D1, D2, P2, P3 |
| (4) | $v_k(A^k) = v_k(A)$ | $\mathrm{PT}^{\mathrm{f}}$ |
| (5) | $v_k(A^k \leftrightarrow A) = T$ | 3,4 |
| (6) | $v_k(A^\tau \leftrightarrow A) = T$ | 2,5 |
| (7) | $\forall k(v_k(A^\tau \leftrightarrow A) = T)$ | 6 |
| (8) | $v_m(A^\tau = A) = T$ | 1,7 □ |

So, $A^\tau$ and $A$ are materially and strictly equivalent.

NON-SYNONYMY OF $A^\tau$ AND $A$. *There are syntactic contexts in which $A^\tau$ and $A$ are not replaceable one by the other without changing the truth-value of the resulting expression: From the sound*

$$\models A^\tau : A^{\tau\varepsilon},$$

*by replacing $A^\tau$ with its equivalent $A$, we receive the unsound*

$$\models A : A^\varepsilon.$$

*Proof.* 1. $\models A^\tau : A^{\tau\varepsilon}$. For every $m$:

| | | |
|---|---|---|
| (1) | $v_m(A^\tau : A^{\tau\varepsilon}) = T \Longleftrightarrow \forall k(v_k(A^\tau \supset A^{\tau\varepsilon}) = T)$ | D3, PC |
| (2) | $v_k(A^\tau \supset A^{\tau\varepsilon}) = T \Longleftrightarrow v_k(A^k \supset A^{k\varepsilon}) = T$ | PT4 |
| (3) | $v_k(A^k \supset A^{k\varepsilon}) = T \Longleftrightarrow$ $(v_k(A^k) = T \Longrightarrow v_k(A^{k\varepsilon}) = T)$ | D1, P2, P3 |

| | | |
|---|---|---|
| (4) | $v_k(A^k) = T \Longrightarrow v_l(A^k) = T$ | PT$^{\text{f}}$ |
| (5) | $v_k(A^k) = T \Longrightarrow \forall l(v_l(A^k) = T)$ | 4 |
| (6) | $v_k(A^k) = T \Longrightarrow v(A^{k\varepsilon}) = T$ | PC |
| (7) | $v_k(A^\tau \supset A^{\tau\varepsilon}) = T$ | 2, (3,6) |
| (8) | $\forall k(v_k(A^\tau \supset A^{\tau\varepsilon}) = T)$ | 7 |
| (9) | $v_m(A^\tau : A^{\tau\varepsilon}) = T$ | 1,8 |

2. Disproof of $A : A^\varepsilon$. We take the following value-stipulation:

$$v_k(A) = T, v_m(A) = F.$$

Then we have

$$v_k(A \supset A^\varepsilon) = F.$$

So, $A : A^\varepsilon$ is not sound. □

To sum up: With the above semantic explanation of the symbol $^\tau$, MacColl's claims concerning the expression "is true" are met: $A$ and $A^\tau$ are strictly equivalent and they are not synonymous in the sense of being replaceable in all contexts.

*3.4. Scope-relatedness*

Despite the seemingly convincing results reached with PT4 so far, there are unusual—and maybe unwanted—features connected with the way in which the soundness of the Gödel-rule is ensured by PT4 along with the hyperintensionality of $^\tau$. With the breakdown of replacability for logically equivalent formulae, there are other unsound formulae and rules one could expect to be sound in accordance with usual extensional or intensional logics. It is not only the replacement rule for logically equivalent expressions, which (as desired) breaks down, but we get into trouble (maybe unwanted by MacColl) with the substitution rule and modus ponens too. E.g., we have the sound (1) along with the unsound (2), which is produced by substitution from (1):

(1) $$\models_{PT4} B \supset (A = A^\tau).$$

(2) $$\not\models_{PT4} B^\tau \supset (A = A^\tau).$$

In addition, we are in trouble with modus ponens: We have $\models_{PT4}$ $(A = A^\tau) \supset (B^\tau \supset (A = A^\tau))$ and $\models_{PT4} A = A^\tau$ as sound formulae, but again we don't have (2) $B^\tau \supset (A = A^\tau)$ as a sound formula.

Connected with the problems concerning the use of modus ponens, the transitivity of implication fails to be generally sound: E.g., we have $\models_{PT4} A^\tau : A^{\tau\varepsilon}$ and $\models_{PT4} A^{\tau\varepsilon} : A^\varepsilon$ , but we do not have $\models_{PT4} A^\tau : A^\varepsilon$.

Such problems emerge, because, according to PT4, formula (2), containing unmodalized occurrences of $^\tau$, has to be evaluated in a valuation $k$ in its substitution (2*), where all $^\tau$ are replaced by $^k$. Because (1) has no unmodalized occurrences of $^\tau$, for the evaluation of (1), according to PT4, no such substitution has to be carried out. So, we get, according to PT4, essentially different formulae, before applying the usual semantic rules as given in subsection 2.3. After applying PT4 we have to evaluate in context $k$:

(1) $$B \supset (A = A^\tau)$$

(2*) $$B^k \supset (A = A^k)$$

and (1) is not equivalent with (2*).

In fact, as treated by PT4, the $^\tau$ in the subformula $A = A^\tau$ of the formula $B^\tau \supset (A = A^\tau)$ has different semantical features compared to the $^\tau$ in a separate formula $A = A^\tau$. This comes from the quantifying-in-power of unmodalized occurrences of $^\tau$ for the whole formula in which such free $^\tau$ occur, with the result that all occurrences of $^\tau$ in such a formula are bound to be *de-re*-evaluated with respect to the given basic valuation. In this sense an unmodalized occurrence of $^\tau$ binds all occurrences of $^\tau$ to the given valuation in which this unmodalized occurrence of $^\tau$ has to be evaluated. However, the formula $A = A^\tau$ is sound only in the *de-dicto*-treatment.

In the PT4-treatment, the scope of the binding power of an unmodalized $^\tau$ is the whole formula in which this unmodalized $^\tau$ occurs. In order to limit the scope of this binding power and to be able to have *de-dicto*- and *de-re*-treated occurrences in the same formula, one can introduce special scope-delimiters '{' and '}', inside which *de-re*-binding of occurrences of $^\tau$ is blocked. Unmodalized occurrences of $^\tau$ will give *de-re*-treatment of $^\tau$ only in formulae scoped by '{' and '}'.

We add to the formula-definition:

If $H$ is a formula, then $\{H\}$ is a formula.

In accordance with the purpose of the scope delimiters '{' and '}', we revise PT4 in the following way:

PT4$^\#$. If there are unmodalized occurrences of $^\tau$ in a formula $H^\#$, then, before applying other semantic rules, the valuation of $H^\#$ has to proceed according to the scheme

$$v_k(H^\#) = T \Longleftrightarrow v_k(H^*) = T,$$

where

(1) $H^{\#}$ is formula $\{H\}$ or formula $H$, and
(2) we obtain $H^*$ from $H$ by replacing all (modalized and unmodalized) occurrences of $^\tau$ outside scoped subformulae $G$ with $^k$.

If there are no occurrences of scope-delimiters in a formula $H$, then

$$\models_{PT4} H \Longleftrightarrow \models_{PT4^{\#}} H,$$

i.e., the treatment of such a formula by PT4$^{\#}$ is just the same as with PT4.

A restricted version of modus ponens then holds in the following form:

MP$^\tau$. $$\models_{PT4^{\#}} G^{\#} \supset H^{\#},\ \models_{PT4^{\#}} G \implies \models_{PT4^{\#}} H,$$

where $G^{\#}$ ($H^{\#}$) is $\{G\}$ ($\{H\}$), if $G$ ($H$) contains unmodalized unscoped occurrences of $^\tau$ and $H$ ($G$) contains modalized unscoped occurrences of $^\tau$.

A restricted substitution rule works with the following conditions:

SR$^\tau$. $$\models_{PT4^{\#}} H \implies \models_{PT4^{\#}} H[p/G]_s,$$

where $H[p/G]_s$ is obtained by substituting all occurrences of the statement-variable $p$ in $H$ by the expression $G$, if $^\tau$ does not occur unscoped in $G$, and by substituting $p$ by the expression $\{G\}$ otherwise.

The transition from PT3 to PT4 allowed not only formulae with *de-re*-occurrences of the truth-predicate $^\tau$ to be handled, but also formulae where all occurrences of $^\tau$ are *de-dicto*-treated. After the introduction of scope delimiters with PT4$^{\#}$ it is possible to handle mixed formulae containing *de-re*- and *de-dicto*-occurrences of $^\tau$, and to express a sound analogue of the Gödel-rule for the truth-predicate $^\tau$,

GR$^\tau$. $$\models_{PT4^{\#}} H \implies \models_{PT4^{\#}} \{H\}^\tau,$$

which without the use of scope-delimiters holds only with the restriction that $H$ contains unmodalized occurrences of $^\tau$.

The main deficiency of the PT3-treatment of MacColl's $^\tau$ was that, because of the unsoundness of the Gödel-rule, MacColl's claim concerning the soundness of strict equivalence between $A$ and $A^\tau$ was not fulfilled even though soundness for material equivalence was. This was so because the truth-predicate $^{\tau}$ was taken in a *de-re*-use in all places of occurrence. Equipped with scope delimiters we can look back to PT3 in order to explicate the claims of MacColl concerning the truth-predicate, and with the following revision of PT3 to PT3$^{\#}$ it is possible to handle *de-re*- and *de-dicto*-occurrences of $^{\tau}$ in the same formula:

PT3$^{\#}$. Before applying other semantic rules, the valuation of $H^{\#}$ has to proceed according to the scheme

$$v_k(H^{\#}) = T \Longleftrightarrow v_k(H^{*}) = T,$$

where

(1) $H^{\#}$ is the formula $\{H\}$ or $H$, and
(2) we obtain $H^{*}$ from $H$ by replacing all (modalized and unmodalized) occurrences of $^{\tau}$ outside of scoped subformulae $G$ with $^{k}$.

Now it is possible to formulate a sound restricted version of the Gödel-rule in the PT3$^{\#}$-treatment:

GR3$^{\#}$. $$\models_{PT3^{\#}} H \implies \models_{PT3^{\#}} \{H\}^{\varepsilon},$$

i.e., if $H$ is sound in PT3$^{\#}$, then the certainty of the *de-dicto*-treated $H$ is sound in the PT3$^{\#}$-treatment.

In accordance with GR3$^{\#}$, the strict implication between $A$ and $A^{\tau}$ is sound in the following form:

$$\models_{PT3^{\#}} \{A \leftrightarrow A^{\tau}\}^{\varepsilon}.$$

Unlike the PT4$^{\#}$-treatment, in the PT3$^{\#}$-treatment we do not have

$$\models \{H\}^{\varepsilon} \implies \models \{H^{\varepsilon}\}.$$

Because of this, we cannot secure *de-dicto*-treatment of $A = A^{\tau}$ by putting the scope-delimiters around it: While $\{A = A^{\tau}\}$ is sound with PT4$^{\#}$ independently of its place of occurrence, $\{A = A^{\tau}\}$ is not sound with PT3$^{\#}$ .

With PT4, PT4$^{\#}$ and PT3$^{\#}$ we have alternatives for the explication of MacColl's intuitive and formal claims concerning the truth-predicate $^{\tau}$. A significant difference between PT4 on the one hand and PT3$^{\#}$ and PT4$^{\#}$ on the other hand consists in the fact that PT4 does not require symbolic tools not found in MacColl 1906.

The crucial point for the explication of MacColl's claims is the fact that the soundness of strict equivalence between $A$ and $A^{\tau}$ presupposes a *de-dicto*-treatment of the truth-predicate $^{\tau}$ inside the formula $A = A^{\tau}$, which is given with every treatment of this formula according to PT4, PT4$^{\#}$ and PT3$^{\#}$. In *de-dicto*-contexts the replacement-rule for sound material and sound strict implications works unrestricted. In the *de-dicto*-use, there is no problem with replacing, e.g., $A$ by $A^{\tau}$ and vice versa. Differently for contexts in which, with the help of $^{\tau}$, local *de-re*-uses of the kind $H^{\tau}$ are constituted, these *de-re*-contexts lose their *de-re*-character at least partially if they are replaced by expressions $G$ without unscoped occurrences of $^{\tau}$, even if $H^{\tau}$ and $G$ are logically equivalent formulae. Despite the logical equivalence between $A$ and $A^{\tau}$, in (1) $A^{\tau} \supset A^{\tau\varepsilon}$, for instance, $A$ in the subformula $A^{\tau\varepsilon}$ is treated *de-re*, while in (2) $A^{\tau} \supset A^{\varepsilon}$, which is produced from (1) by the replacement of $A^{\tau}$ by the logically equivalent $A$, the $A$ in $A^{\varepsilon}$ is treated *de-dicto*. This is the reason why (1) is sound and (2) is not, even though (3) $A = A^{\tau}$ is.

With the truth-operator $^{\tau}$, we have an extremely instructive example of the context-sensitivity of MacColl's symbolic language and the importance of context for the determination of features of the signs in his formalism, one which has strong non-classical consequences. The kinds of explicit differentiation between *de-re*- and *de-dicto*-occurrences of $^{\tau}$ introduced make it possible to give explicit characterizations of the difference between its *de-re* and *de-dicto* uses, a distinction which is hidden in *Symbolic Logic*. This demonstrates that MacColl was not only the pioneer of modern modal logic. With his truth-predicate, MacColl introduced a sophisticated tool into his logical framework which finds its logical foundation not in ambiguities but in the aim to capture special context-sensitive logical features of the way in which modal statements and statements in modal contexts are evaluated *de re* or *de dicto* in natural and formal languages.

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