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MACCOLL AND MANY-VALUED LOGIC: AN EXCLUSIVE CONJUNCTION

1. Rescher's Statement

In his valuable compendium *Many-Valued Logic* Nicholas Rescher states (1969, p. 4) that the founding fathers of many-valued logic prior to Lukasiewicz are Charles Sanders Peirce, Nicolai Vasil'ev and Hugh MacColl. This paper shows very simply, against Rescher, that MacColl's logic cannot reasonably be counted as many-valued.

2. Many-Valued Logic, What

A logic can be given a many-valued semantics and still not be *essentially* many-valued. Here is an example. A propositional logic C2 based on negation and conjunction is given a semantics with the following four-valued truth-tables (conjunction on the right):

	$\neg$	1	2	3	4
$*1$	4	1	2	3	4
2	3	2	2	4	4
3	2	3	4	3	4
4	1	4	4	4	4

The reason C2 is not essentially many-valued is that its tautologies and valid deductions coincide with those of classical two-valued logic C: the matrices of C2 are simply the Cartesian product of the bivalent matrices of classical logic with themselves. By contrast, if one adds a further connective Γ with the matrix

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	Γ
∗1	2
2	2
3	4
4	4

then the resulting logic is essentially many-valued: it is in this case equivalent to Lukasiewicz's last many-valued and intendedly modal logic L (1953). A similarly essentially many-valued system is Lukasiewicz's first trivalent logic, which is poorer in tautologies than classical logic.

I say that a logical system L is *essentially many-valued* when any semantics with respect to which L is sound and complete is such that:

MV1 the semantic values or statuses of its sentences (closed wffs) include both true (T or 1) and false (F or 0) and at least one other value besides, distinct from T and F.

MV2 having the values T, F and any of the others are *pairwise exclusive* and *jointly exhaustive* (PEJE) of the semantic statuses of any sentence on any given valuation of sentences.

MV3 the connectives of L are *value-functional*, that is, for any connective K and any sentences $S_1, \ldots S_n$, the value of $K(S_1, \ldots, S_n)$ under a given interpretation I, which we write $|K(S_1, \ldots, S_n)|_I$, is a function of $|S_1|_I, \ldots, |S_n|_I$ alone, as determined by the fixed interpretation of K.

In the case where a logic has higher-order operators such as quantifiers the analogous principle to MV3 applies:

MV4 the value of a sentence containing an operator as main symbol is a value-function of the values of its instantiations.

It follows from these conditions that the tautologies and valid inferences of L do not coincide with those of classical logic, for if they did it could be given a bivalent semantics.

3. Why MacColl's Logic is Not Many-Valued

At first sight, the statuses of propositions in MacColl's logic make it look as though one can support the contention that his logic is essentially many-valued. In the definitive statement of his views in 'La logique symbolique' (1901, p. 138), he introduces five semantic values for propositions, giving them these glosses:

$$\tau - \text{true}, \iota - \text{false}, \epsilon - \text{certain}, \eta - \text{impossible}, \theta - \text{variable}$$

and on pp. 140-142 he asserts a number of equations linking them and their associated single-place connectives A^τ A^ι A^ϵ A^θ A^η (A is true, false, certain, variable, impossible):

$$\epsilon^\eta = \epsilon^\theta = \theta^\epsilon = \theta^\eta = \eta^\epsilon = \epsilon^\iota = \eta$$
$$\eta^\eta = \epsilon^\iota = \eta^\iota = \epsilon^\tau = \epsilon$$

All five values together cannot give a 5-valued semantics because τ and ι are PEJE: MacColl asserts that $A^\tau + A^\iota$ (where '+' stands for disjunction) and $(A^\tau A^\iota)^\eta$ (where juxtaposition stands for conjunction). They are clearly simply the two classical values so could not be considered unless added to others, which is ruled out by these principles. The three values ϵ, θ and η likewise but more promisingly form a PEJE set because

$$A^\epsilon + A^\theta + A^\eta, (A^\epsilon A^\eta)^\eta, (A^\epsilon A^\theta)^\eta, (A^\theta A^\eta)^\eta.$$

MacColl defines a strict implication connective: '$A : B$' is read as 'If A then B' and understood as synonymous with 'it is impossible that A and not B' or $(AB')^\eta$ where B' is the negation of B and defined as synonymous with B^ι. He affirms these implications

$$A^\epsilon : A^\tau \qquad A^\eta : A^\iota$$

the first being akin to the modal formula T. Since on p. 144 formula (10) MacColl also affirms that $A^\epsilon = (A = \epsilon)$ presumably also

$$\epsilon^\epsilon = \theta^\theta = \epsilon \text{ and } \theta^\eta = \eta$$

so since $\tau^\tau = \epsilon$ and $\iota^\tau = \eta$ (for as terms and factors τ and ι are said by MacColl to be equivalent respectively to ϵ and η—p. 140 ftn. 3), in general

$$\alpha^\beta = \epsilon \text{ if } \alpha = \beta$$
$$\alpha^\beta = \eta \text{ if } \alpha \neq \beta$$

and the following seemingly value-functional connectives seem to emerge (the last two values for A^θ being as conjectured):

A	A^ϵ	A^θ	A^η	A^τ	A^ι
$*\epsilon$	ϵ	η	η	ϵ	η
θ	η	ϵ	η	θ	θ
η	η	η	ϵ	η	ϵ

The connective A^ϵ looks then like a "strong assertion" functor: it gives the strongly designated value ϵ when its argument has value ϵ and the strongly anti-designated value η otherwise.

But ϵ, θ and η do not sustain a value-functional semantics because the implication connective written ':' has an incomplete matrix with respect to the values as follows

	$A : B$	A: ϵ	θ	η
B	$*\epsilon$	ϵ	η	η
	θ	ϵ		η
	η	ϵ	ϵ	ϵ

$|A : B|$ can be anything for $|A| = \theta = |B|$. Suppose A is 'It is raining' then for $B = A, |A : B| = \epsilon, |A : B'| = \eta$, while if B is 'It is Wednesday', independent of $A, |A : B| = \theta$.

Similarly, the matrix for conjunction is incomplete:

	AB	A: ϵ	θ	η
B	$*\epsilon$	ϵ	θ	η
	θ	θ		η
	η	η	η	η

Hence neither the three nor all five values on offer provide a value-functional semantics for the important implication and conjunction connectives.

4. MacColl's as a Modal Probability Logic

The intended and stated intepretations of A^ϵ, A^η and A^θ are as probability propositions

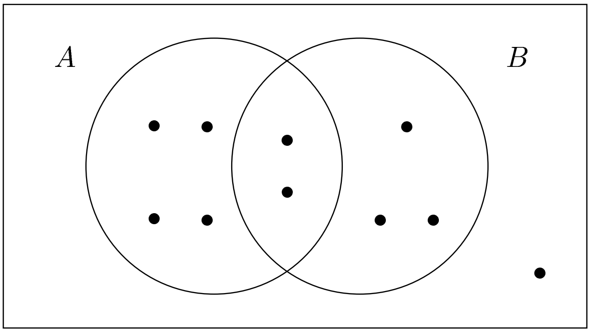
$$\begin{array}{ll} A^\epsilon & P(A) = 1 \\ A^\eta & P(A) = 0 \\ A^\theta & 0 < P(A) < 1 \end{array}$$

where MacColl distinguished between formal and material necessity (certainty), impossibility and variability. The uncertainty attaching to the interpretation of some of MacColl's constants and connectives supports Russell's contention (1906, p. 256 f.) that MacColl fails clearly to distinguish between propositions and propositional functions. Nevertheless, with some charity (which Russell was unwilling to dispense to a proponent of modal logic) MacColl's general intentions are clear enough as outlined above.

If we interpret the three functors as above, then the lack of value-functionality is immediately explained: the probability function P is not value-functional: $P(AB)$ is not a function of $P(A)$ and $P(B)$, but satisfies merely the inequality $0 \leq P(AB) \leq \min\{P(A), P(B)\}$. If $P(A) = 0.4$ then $P(A') = 0.6$ and $P(AA') = 0$. The summation law for probability

$$P(A) + P(B) = P(A + B) + P(AB)$$

was known to MacColl: it follows from the Kolmogorov axioms. If we take A and B as finite sets given by the Venn diagram below, where $P(X)$ gives the probability that a dot chosen at random is within the area X and $P(X')$ gives the probability that such a dot is in the complementary area to X,



then here $P(A) = 0.6$, $P(B) = 0.5$, $P(A+B) = 0.9$, $P(AB) = 0.2$ and $P(A/B) = P(AB)/P(B)$, here 1/3, again a fact known to MacColl (cf. 1901, p. 154).

Because the intended and actual application of MacColl's logic is to probabilities, then despite there being many "values" for propositions it is not value-*functional*, so it is misleading to regard MacColl's logic as essentially many-valued. Rather it is a modal logic of probability, which is not fully value-functional. It is true that not just in MacColl's day but for some time afterwards, logicians such as Lukasiewicz did not clearly distinguish probability logic from many-valued logic. In his *Grundlagen der Wahrscheinlichkeitsrechnung* of 1913, and even after developing many-valued logic as such, Lukasiewicz still tends to run the two together (as in Lukasiewicz 1930, cf. Lukasiewicz 1970, p. 173). Since the point of a many-valued system is to interpret the logical constants in a way analogously with that of bivalent logic, that point is lost if value-functionality goes. Instead one is dealing as here with a modal rather than a many-valued system, even if we use a plurality (greater than two) of other "statuses", as e.g. when talking about truth "at" several different possible worlds in the standard semantics for modal logic.

Ironically, a philosopher whose views on the several values a proposition may have (including others apart from true and false) were also forged in conjunction with a theory of probability was Alexius Meinong, whose work was influential on Lukasiewicz. As I have shown elsewhere (Simons 1989), Meinong's clear affirmation of values for propositions other than the two classical ones makes him, though not himself a logician, a precursor of Lukasiewicz's work and a founding father of many-valued logic with greater title to this status than MacColl.

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