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CONNEXIVE LOGIC

Dedicated to Prof. Dr. Christian Thiel
on the occasion of his 60th birthday.

Initially, the paper proposes a fallibilistic motivation for logical systems containing Aristotle's thesis and strong or weak versions of other connexive theorems. In essence a pragmatic account of implication is proposed: p implies q if, and only if, q *follows* from p.

Subsequently, the article introduces a sequent calculus LKS of classical propositional logic that is free of the paradoxes of logical implication. Primarily, the system ensues from an exhaustive control of thinning in terms of structural rules as well as rules for logical constants. It is possible to enrich LKS by rules for the introduction of connexive implication. For the resulting system LKK a consistency proof is readily at hand. Hence, LKK does not contain those varieties of connexive logics Routley and Montgomery have shown to be inconsistent.

The concept of a connexive implication has been known at least since the times of the Stoic academy. It may be of Aristotelian origin.[1] Despite its long history, neither in contemporary logic nor in analytic philosophy has this concept been studied attentively.[2] In their critical esteem for connexive logics Routley et al. (1982) point out that philosophical inquiries into their sense have been neglected. On most occasions these systems were developed as formal means by which to avoid deficiencies of classical or strict implication.[3] In particular, a logical system in which for example $\neg(a \supset \neg a)$ and $(a \supset b) \supset \neg(a \supset \neg b)$

[1]For the concept's ancient history, cf. Bocheński 1956, 20.09 seq.; Hülser 1987, §4.4.3.1; Mates 1973, ch. 5, §1; Routley et al. 1982, p. 82 seq. For connexive implication in medieval logic, cf. Read 1993.

[2]For discussions of connexive principles during the last decades, cf. in particular Strawson 1952, Nelson 1930, Angell 1962, McCall 1966, Routley and Montgomery 1968, and Astroh 1993a.

[3]Routley et al. 1982, p. 87.

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are valid schemes[4] allows for a reconstruction of categorical syllogistics independently of existential presuppositions.[5] Generally speaking, logics of this kind avoid various so-called paradoxes of implication.[6] To the extent that they do they can include schemes of the above-mentioned kind.

The present article sums up a theory of logical connectivity that accounts for the concept's far-reaching philosophical impact.[7] In essence, the theory conceives of an implication between A and B as a relation which holds good if, and only if, B *follows* from A. Rules of inference are methods of rational orientation such that series of premises allow one to reach *specific* conclusions. A formal presentation of such rules recasts a semiotic structure in which, on the basis of unquestioned presuppositions, a *goal* is set. For this reason, a system of connexive logic can serve as a basis for philosophical logics that claim to account for formal properties of rational orientation. In this sense, especially, epistemic logics contribute to a formal understanding of human orientation. Hence, it may seem appropriate to set forth a connexive system of logic as nothing but a formal theory of logical intentionality.

By necessity, any alignment with a goal is bound to presuppositions. In this very orientation they cannot be problematic. For if they were, the orientation's original points of reference could not matter as such. In this sense, for instance, an assertion's subject term specifies the referential presupposition of a given predication. In general, predication is assumed to pertain to a range of objects. The relevant predicate term is meant to apply to these items. An assertion's subject term thus fulfils a preparatory task. It fixes the referential context on which a judgement's sense depends. Connexive logics, however, allows for a reconstruction of categorical syllogistics in terms of propositional logic that does not presuppose existential import. This confirms that a logic of intentionality is in fact a logic of presupposition.[8]

The present theory of connectivity thus offers an account of elementary forms of rationality, but it does not rely on a theory of consciousness. It is meant to motivate and to provide a method for an accurate presentation of logical connectivity. However, instead of a straightforward rendering of a connexive system of logical inference, a

[4]In the literature they are called *Aristoteles* and *Boethius*, respectively. For a detailed account of *Boethius* cf. Read 1993 and Pizzi and Williamson 1997. '$\supset$' is used informally in the present section.

[5]Cf. MacColl 1878, McCall 1966 and Astroh 1996.

[6]Cf. Routley et al. 1982, p. 82.

[7]A detailed justification, presentation and application of the logic introduced in this article will be given in Astroh 1999.

[8]Cf. Strawson 1952, p. 176.

reformulation of classical propositional logic will lead to the intended goal. In order to achieve it, the language of a classical sequent calculus will be extended by a system of indication. At each stage of its application a set of indicators will show the extent to which a deducible sequent articulates an intentional structure—and thus may count as a genuine representation of an inference rule.

1. Orientation, Inference, and Implication

In his seminal investigation "Über das logische Schließen" (1934–1935) Gentzen refers to the classical sequent calculus LK as a *Schlußweisenkalkül*, i.e. a calculus of inference figures. In a sense this view is unproblematic—on the condition that one may conceive of an inference figure as nothing but the syntactic counterpart of a classical logical consequence as defined in Tarski's formal semantics.

At first glance, however, the suggested understanding of sequents of the form $A_1, \dots, A_n \rightarrow B_1, \dots, B_m$ derivable in Gentzen's system LK conflicts with our intuitive grasp of logical rules. This informal account of a classical sequent calculus naturally is at odds with the so-called paradoxes of logical implication that LK contains. For the sake of brevity, their discussion will be omitted here.[9] Even so, the subsequent inquiry will set out from the assumption that their inclusion in LK results from a uniform reason: The system provides no information as to whether and how a sequent's components effectively contribute to its being an inference figure. The subsequent account of logical connectivity will ensue from a systematic assessment of this contribution and of its articulation within a classical sequent calculus. In this way an elementary *calculus of logical inference* will be developed. It will still contain the familiar paradoxes, but likewise it will include means for their systematic identification. In essence, the intended system will result from a systematic assessment of LK's structural features: interchange, contraction, and foremost thinning. Not just their explicit articulation in LK's structural rules must be accounted for, but, in particular, the implicit occurrence of thinning within rules for logical constants.

A sequent calculus will not count as a genuine calculus of logical inference unless all thinning has to result from an explicit introduction of formulae, i.e. from the application of a structural rule. Subsequently, a philosophical account of thinning will be developed conforming to the

[9] Cf. Astroh 1999.

intended, informal understanding of a sequent calculus. Accordingly, a system LKA equivalent to the propositional part of LK will be proposed.

In LKA, thinning is not possible unless it results from an application of appropriate structural rules. Hence, the rules for the introduction of disjunction or implication in the antecedent or of conjunction in the succedent will have it that their premises may differ *at most* in their side formulae, and that in their conclusion all other schematic expressions occur but once. Likewise, the rules of LKA for the introduction of disjunction or implication in the succedent, or of conjunction in the antecedent, lay down that the conclusions of these rules contain exactly the same formulae as their premises, except that the introduction of a logical constant has turned two of them into components of just one new item. *Explicit* thinning, then, is the only means by which to enrich a sequent with further schematic expressions for propositions.

There are only two reasons why a sequent derivable in LKA will *not* count immediately as a rule of inference. Either preceding steps in its derivation, i.e. applications of thinning, have introduced irrelevant propositions such that by inserting a logical constant, a sequent lacking these formulae is derivable, for instance by introducing conjunction in the antecedent; or a rule of LKA permits all by itself the introduction of a logical constant such that the resulting sequent cannot stand for an inference rule.

Independently of thinning the mere introduction of implication in the antecedent allows for the derivation of sequents that could not count as rules of logical inference. For example:

(1) $a, a \supset \neg a \rightarrow$

(2) $a \supset \neg a \rightarrow \neg a$

(3) $\neg a \supset a \rightarrow a$

(4) $\neg a, \neg a \supset a \rightarrow$

It is thus prerequisite to question the sense in which each of the propositional connectives contributes to a logic of inference. Their unconditional application will not be justified unless the relevant calculus allows translation of an inferential relationship between propositions into a logical relationship between corresponding propositions.

These conjectures will lead to a truly inferential understanding of sequents $\Gamma \rightarrow \Theta$. However, for this purpose, their syntactic design needs a detailed justification. Gentzen's reasons for choosing this presentation of logical form were entirely strategic. The sequential rendering of predicate logic allowed for a most promising proof of its syntactic consistency. It became the starting point of comprehensive, metalogical investigations. The present task, however, requires an elementary, pragmatic sense of a sequent $A_1, \dots, A_m \to B_1, \dots, B_n$ to be established so that antecedent and succedent may count as series of premises and of partial conclusions, respectively. Subsequently, sequents of the relevant kind will present themselves as schemes of basic *forms of orientation.* In particular, the logical constants contained in its premises $A_1, \dots, A_m$ or in its partial conclusions $B_1, \dots, B_n$ will serve as indicators of this form of orientation. Therefore, the subsequent, formal rendering of a connexive logic is not an arbitrary presentation thereof—independently of the system's philosophical motivation. It will, on the contrary, result from an *epistemic* account of inferring. In essence it will stem from a semiotic conception of information processing. The syntactic design of a sequent will present itself as a uniform articulation of the fallibilistic features included in the guiding concept of inference.[10]

Inferences exhibit a particular variety of orientation. Presupposed states of affairs which as such are not liable to any inspection pertain to our actions and thoughts primarily through their consequences. For if a state of affairs was independent of any rule-governed relationship to further indicated information, it could not matter what that state of affairs consisted in.

Again, the inspection of an assessed state of affairs amounts to nothing but a rule-governed passage to further information. It decides upon the value of the former in an objective and, in principle, reproducible manner. If, for instance, someone wants to refute or confirm what he is told he will apply rules appropriate to achieve the intended assessment. The information thus received will at least in some cases determine the value of his original information. The assessment might consist, for instance, in a series of perceptions or in the steps of a proof. At any rate, it will consist in a sequent of transitions that *count* as successful or unsuccessful contributions to an, in most cases, indefinite development.

An agent's ability to orient himself thus consists in a particular disposition to process information in a rule-governed form. Specifically, orientation consists in any *adaptable* form of information processing. The intended, informal justification of sequents as inference figures will result from subsequent comments on this definition.

[10]The present strategy does not violate the fruitful maxim that the philosophical justification of a logical system should not depend on a particular variety of calculus. For the principle by no means says that all equivalent calculi exhibit a system's sense equally clearly and distinctly. In the present case sequent calculi are favoured for heuristic reasons.

1.1. Information Processing

While the goal of the present investigation is not to define information or its processing, certain comments about the concept are still called for. Various empirical sciences investigate the way in which information is processed. For the present inquiry, only those logical conditions which are presupposed in any form of information processing, and *a fortiori* its description, may be relevant. Hence, the information an agent processes will be referred to under deliberate restrictions: Either it consists in a state of affairs that can be described in some language, or it is at least possible to describe a state of affairs that exists if, and only if, the relevant information is processed. For instance, a lower animal's reaction to a stimulus will likely not consist in an orientation with reference to a state of affairs. But even scientific research into its behaviour cannot avoid any description of states of affairs that exist if, and only if, it processes the information it gains. How an agent really relates to the information whose processing a description ascribes to him does not matter in the present context. The experience of a being with which we interact, but do not communicate, is inaccessible. Subsequent considerations will thus refer to information as *propositional* information in the proposed weak sense.[11]

No one can exclude that some forms of information processing are beyond our ability to describe—this can be due to a lack of empirical knowledge or result from the absence of appropriate means of representation. The present—in a weak sense—propositional account of information reckons with this possibility. The present stance does not imply that man is a measure of all things. It rather acknowledges that he is a measure of his own ignorance.

As any other form of information processing, orientation is an ordered form of behaviour. In the first instance, one therefore has to comment on the kind of rules an agent follows when processing information. Naturally, their application consists in gathering information under the constraint of acquired information. The present investigation is not concerned with empirical circumstances under which an agent manages to process information. In what follows, only a *form* of ordered information processing is examined.

If an agent was not concerned about the success of his applying rules it would not make sense to say that he was following them. An agent orienting himself must be able to receive information in such a way that he accepts its progressive reception as a result of his very behaviour.

[11] The underlying philosophical concept of information will be discussed in Astroh 1999. For a thorough account of logical investigations into the concept of information cf. Dunn 1999.

He has to dispose of a measure of its success *before* it comes about. Otherwise there is no situation whose realization he will instantly experience as a successful or disappointing effect of his behaviour. In this sense an agent's orientation could, for instance, depend on whether it is getting cooler. His experience could tell him that with increasing darkness it gets cooler. In this case he must be able to behave in such a way that if it is getting darker he will realize that it is getting cooler. Otherwise, he cannot rely on an according empirical rule. Information cannot influence his behaviour unless he receives it in connection with further information. However, in spite of their connection he has to receive them in *different* ways. Otherwise a relationship between these pieces of information cannot pertain to his behaviour. If he perceives, for instance, that it is getting both darker *and* cooler, then he cannot relate to an increase in darkness as a measure of a cooling-down. There is no pertinent linkage between these pieces of information unless the different ways in which he is entertaining them coherently relate to one another. Otherwise their connection cannot shape a behaviour that amounts to a progressive, more or less successful reception of information. In the simplest case a connected reception of information will consist merely in a reception of information which is due to an acquisition of some other information. For instance, one could perceive that it is getting darker, and by receiving this information one could come to know of a cooling-down, although one does not equally perceive it. Hence, an *immediate* reception of information includes an indirect reception of some other information. In this elementary case an item of information is received as an *indication* of another one.

Indications are essential to an agent's behaviour. They reach beyond his immediate knowledge so that his subsequent reception of information can comply with his previous acquisitions. In the simplest case, such a form of behaviour consists in an efficiently ordered information processing: Indicated information becomes relevant in that its indirect reception allows for the application of an according rule of behaviour. If it is adequate and applied appropriately, the corresponding behaviour will turn the indicated piece of information into one that is received all by itself. And, being received as such, it will indicate further information.

A piece of information can indicate several other ones, logically independent of one another. In this case the agent is bound to *choose* among alternative varieties of behaviour. For instance, the state of affairs that it is getting darker could indicate to him where it is cooling down and where it becomes warmer. In such circumstances he will bring about an immediate reception of the one and avoid an instant experience of the other. Because of his choice he might succeed in ruling his condition for his own benefit.

In the present perspective, information processing consists in a rule-governed behaviour by which the immediate reception of indicated information is achieved or avoided. Behaviour of this sort cannot be efficient unless the indicative connection is due to the involved pieces of information. In other words, a piece of information a will not indicate a piece of information b unless one may *infer* b from a. Otherwise one could not rely on an indication of information in order to process it effectively. The relevant relationship between indication and material inference can be rendered schematically:

(5) $a\phi b \rightarrow a \sqsupset b$

'a' and 'b' are schematic expressions and stand for terms for elementary states of affairs. 'ϕ' and '$\sqsupset$' stand for expressions for relations between states of affairs. Hence, the schematic expressions '$a\phi b$' and '$a \sqsupset b$' stand for terms for according states of affairs: $a\phi b$ is the case if, and only if, the state of affairs a indicates the state of affairs b. $a \sqsupset b$ is the case if, and only if, one may infer the assertion of b from the assertion of a. '$\rightarrow$' indicates a *logical* inference rule. Subsequently '$\Rightarrow$' will indicate a *material* inference rule.

Indication presupposes that one may infer an indicated piece of information from the one indicating it. On the one hand, an accurate understanding of information processing and orientation therefore requires a definition of the presupposed concept of logical inference. On the other hand, inference figures are relevant only as constitutive aspects of forms of empirical orientation. Therefore, the informal justification of a concept of logical order has to result from a comment on the form of orientation presupposing it. In this respect the subsequent discussion of orientation as *adaptable information processing* will justify a sequent calculus as a formal articulation of a classical logic of inference.

1.2. Adaptable Behaviour

The preceding, semiotic, description set out from successful information processing. Naturally, it has never been presupposed that processing of information could not fail. Elementary rules of informed behaviour are applied unceasingly, and thus confirm their unquestioned value. However, despite one's appropriate behaviour one can fail to pass from a putative indication to its immediate confirmation. Increasing darkness might be taken to indicate, given normal circumstances, a cooling-down. But, presumably, there are exceptions to this default case. Under particular circumstances the temperature does not change in spite of an obvious increase in darkness.

In some such cases one might have to give up a formerly undisputed rule altogether. Life's authority could demand a radically different form of adaptation. However, such fundamental changes are not the rule. Usually, deviations amount to occasional, unproblematic irritations. Instead of a basic correction, a refined adaptation will account for such secondary, perhaps temporary, experiences.

In the simplest case one will learn to reckon in advance with occasional exceptions. An immediate reception of an information a will now count as an indication which normally points to an information b, but occasionally anticipates the experience of an information c. Such a refined indication where a points to b as well as c does not presuppose an inference of b from a or of c from a. In view of a one can merely conclude b, *or else* c:

(6) $\quad a \Rightarrow b, c$

The scheme says that from a follows b—or else c. The assertion of a now presents itself in an inferential relationship to both an assertion of b and an assertion of c. In this respect (6) describes a material rule of inference. It says that a premise, i.e. the assertion of a, *partially* admits of a conclusion b and partially admits of a conclusion c. The order in which those partial conclusions are mentioned indicates to what extent concluded information deviates from a given standard. This disposition mirrors an order in which concessive modifications of an original rule have been gained. For elementary pieces of information $a, b_1, \ldots, b_n$ the form of a *concessive rule of inference* presents itself as follows:

(7) $\quad a \Rightarrow b_1, \ldots, b_n$

The present, fallibilistic account of elementary orientation thus motivates to let the succedent of a classical sequent consist of a series of propositions. In this horizon a succession of partial conclusions presents itself as a constitutive aspect of a genuine form of orientation.

In the order of a sequent's antecedent a similar sense can be accounted for: An adaptation which concedes exceptions amounts to a rather limited revision of a rule of orientation. Here learning merely consists in an exemplary recognition of circumstances that may come about. Not the information on which an agent's behaviour is focusing, but the state of affairs from which it sets out remains invariant. The agent does not refine the indicating state of affairs on which a well-established range of alternative forms of behaviour might depend.

If, however, he succeeded in gaining a more subtle grasp of the initial situation he could, perhaps, preserve his former orientation, i.e. the scope of his original alternatives. For instance, in response to the experience that an increase in darkness does not always lead to a cooling-down, but that occasionally the temperature does not change, one could learn to notice that darkness grows in various ways and under different circumstances. In the end one might grasp in which particular cases increasing darkness indicates a cooling-down. Instead of altering the sense of one's own behaviour by concessions, one learns to differentiate the context in which one's orientation matters. In this way an agent can preserve the original perspective of his behaviour. By an improved acquaintance with actually divergent situations he gradually succeeds in handling them.

Again, an established indication is modified. But instead of accepting a failure as a possible exception, a genuine revision of the rule takes place. Instead of conceding exceptions, such as in (6) and (7), the rule's scope of application undergoes a *constructive restriction*. Here,

(8) $a \Rightarrow b$

is replaced by

(9) $a, c \Rightarrow b$

Hence in

(10) $a, c_1, \ldots, c_n \Rightarrow b$

the disposition of restrictions c_ν mirrors the constructive development of an elementary form of orientation. Again, the succession of these refinements is not irrelevant. Possibly a restriction $c_{\nu'}$ will not be singled out unless its predecessor c_ν has been established. The present investigation does not afford examination of the constructive sense of a revised orientation in any more detailed manner. At present, only an informal introduction of classical sequents is at stake. It is meant to present them as articulations of inference rules. In the preceding passages the basic form of adaptable information processing has been described in terms of a primary, empirical differentiation. The difference between concessive and constructive modifications of an inference rule tacitly points to the different sense of classical and constructive logic. This subject must be left out in the following. As here a philosophical justification and formal presentation of a connexive propositional logic is at stake, it is more urgent to deal with other matters. The following section will show in what way the sense of an elementary inference figure can motivate both a classical propositional logic LKA and its modification to a connexive system LKK.

2. A Fallibilistic Justification of Classical Propositional Logic

Material inference rules are a constitutive aspect of factual orientation. The sequential scheme

(11) $a_1, \ldots, a_m \Rightarrow b_1, \ldots, b_n$

articulates this informal insight. Rules of this kind do not hold unless there are formal rules of orientation, too. In the first instance, logical relations between various pieces of information depend on the range of inference rules to which they are jointly subjected. However, the use of logical connectives allows articulation of logical relations not just *between*, but likewise *within*, assertions. Their introduction will result in a systematic account of formal, and, in particular, of logical inference rules. The present section is meant to justify this strictly inferential understanding of logic.

Above, the concept of an inference rule was discussed in terms of epistemic conditions for its application. In this way reasons for a progressive differentiation of such rules could be proposed. This fallibilistic conception of inference figures will now lead to a justification of classical propositional logic and its connexive variant. For this purpose the subsequent argumentation will rely on the following conventions for the presentation of a logical calculus. In fact they have been applied tacitly in previous sections of this article.

2.1. Conventions

Capital Greek letters stand for possibly empty series of schematic expressions for composite terms—the latter standing for separate pieces of propositional information. In the schematic presentation of an inference rule, some, though not all, sections of its succedent or antecedent may be empty.

Capital Latin letters are used as schematic expressions for composite terms standing for propositional information. The use of brackets guarantees their unambiguous composition. Small Latin letters serve as schematic expressions for primitive terms standing for propositional information. Owing to their role, Latin letters are called *propositional schemes.*

Letters of these kinds immediately following one another are separated by commas. If necessary, schematic letters are distinguished with the help of numeric subscripts.

If Θ and Γ are well-formed series of schematic expressions, then $\Theta \rightarrow \Gamma$ stands for a logical inference rule. Its syntactic presentation is called a logical sequent. The schematic expressions on the left of the arrow form its antecedent; those on its right form its succedent. Antecedent and succedent indicate an inference rule's premises and partial consequences, respectively.

The expressions '$\wedge$', '$\vee$', '$\supset$', '$\sqsupset$' and '$\neg$' refer to terms for conjunction, disjunction, subjunction, connexive subjunction and negation in an according logic of inference. If A and B are terms for propositional information, then $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$, $(A \sqsupset B)$ and $\neg(A)$ are so too. Where misunderstandings are unlikely brackets may be omitted.

2.2. A Logical Criterion for Assertible Information

In previous sections the basic form of a material inference rule was established. It consists in a classical sequent of elementary assertions. The leading, fallibilistic account of this inferential form is sufficient for a complete, inductive justification of a classical system of inferential forms. For this purpose, materially justified inference figures must be distinguished from those that are formally valid:

D 1.1 An inference figure is *materially valid* if, and only if, in some cases its application can call for a concessive or constructive revision.

D 1.2 An inference figure is *formally valid* if, and only if, there is no case in which its application can call for a revision.

The system LKA gives an inductive definition of the classical inference figures as formally valid. The following considerations restrict themselves to a systematic presentation of *propositional* inference figures.

These introductory remarks on the pragmatic sense of classical sequents tacitly assume that elementary propositional information is at any rate *assertible* information. There are at least two reasons for this assumption: On the one hand, elementary pieces of information are gained and processed in an unquestioned practice. The primitive examples of the previous section were meant to indicate this methodological presupposition. In the first instance, information is thus processed in a natural manner. This elementary practice cannot offer any criteria for the identification of propositional information that is not assertible.

On the other hand, it would lead beyond the scope of this article to propose any account of an elementary assertion's internal structure, i.e. a theory of predication. A theory of this sort would set out from the aforementioned practice of sharing an elementary form of life. Only with reference to this basic condition of all joint orientation is it possible to establish criteria as to what one can reasonably assert about objects of particular sorts.

Undoubtedly, such a theory of predication should stem from the same fallibilistic semiotics which at present leads to a connexive conception of inference rules. For now, however, it will do to account for an information's assertibility in terms of an inference figure. Subsequent considerations will not concern an assertion's predicative structure. Hence, the following definition of assertibility will turn out to be fruitful:

D 2 A propositional piece of information A is *assertible* if, and only if, there is a premise from which the assertion of A follows, as well as a conclusion following from it, and if this very assertion is itself neither a premise nor a conclusion of any arbitrary assertion.

Subsequent considerations will set out from the fact that contradictory and tautological pieces of information are not assertible.

Formal inferences no less than material ones matter with regard to their content. Only their validity is independent thereof. Inferences of the first kind merely render explicit on what grounds inferences of the second kind are justified. *Modus tollendo ponens* clearly illustrates this relationship between material and formal varieties of inference. As the following argument shows, even the totally inefficient, though not entirely irrelevant, inference figure $a \to a$ matters with regard to its content.

On the one hand, it is, of course, a logically relevant scheme. The design of sequent- or tableaux-calculi clearly shows in what way the elementary scheme $a \to a$ helps to justify complex rules of logical inference: The conclusion says nothing more than what the premise assumes. Revisions are thus excluded.

However, in view of the above definition $a \to a$ also says that at least in a formal sense a is a piece of *assertible* information. This basic quality of some inference figures $A \to A$ is essential to the subsequent, philosophical argument for a connexive propositional logic LKK, because the justification of its rules will result from the present, fallibilistic justification of LKA. All its rules will be set forth as rules for the derivation of classical inference figures—immune to any kind of revision. In this pragmatic respect, LKA's initial sequent $a \to a$ cannot by itself count as a genuine inference figure. However, it may be taken to articulate the absolute assertibility of a piece of information a—a quality which any inference figure involving a will presuppose. In the sequel, this concept of assertibility will account for LKA's gradual transformation into a connexive system LKK. The system LKA grants an assertion's absolute or logical assertibility for any elementary information a, though not for any logically complex piece of information A. But then one may not conceive of all sequents derivable in LKA as authentic articulations of inference rules. LKK will have to compensate for this deficiency. For every piece of propositional information A mentioned in a derivable sequent, the calculus will show whether it is assertible. Its strictly pragmatic presentation of classical logic will, in other words, indicate the way in which each piece of propositional information contributes to a given inference figure. And, on the basis of this structural information, rules for the introduction of a connexive implication will present themselves.

Only a well-developed, sufficiently complex theory of predication could restrict the present assumption that all elementary propositional information is absolutely assertible. As long as such a theory is lacking, nothing but the applicability of inference figures can characterize assertibility. The required theory certainly would restrict the domain of assertible information, though it could not invalidate the present, external criterion. With this reservation it is admissible to introduce a propositional concept of connectivity. The outline of a connexive predicate logic could at most restrict the presupposed domain of elementary assertible information.

Certainly, there is no point in inferring from an assertion of a this very assertion. But the inference figure $a \rightarrow a$ also articulates that there is at least one assertion from which the assertion of a follows and which follows from this assertion—while neither $a \rightarrow \Gamma$ nor $\Theta \rightarrow a$ are derivable.

An answer to the question what can be predicated, and *a fortiori* what can be asserted, is neither an empirical nor a contingent matter. For the applicability of logical inference rules may not depend on any *particular* kind of information. Assertibility is a constitutive aspect of an advanced variety of information processing. It determines the *form* in which information contributes to orientation and communication.

Not all sequents $A \rightarrow A$ derivable in LKA will inherit this sense of $a \rightarrow a$. However, the design of a connexive calculus requires a method which for every proposition offers a criterion for its assertibility. The formal presentation of a connexive system will thus depend on a restriction of LKA such that together with a derivable sequent $A \rightarrow A$ neither $A \rightarrow \Gamma$ nor $\Theta \rightarrow A$ is derivable.

The present justification of classical and, finally, of connexive logic does not depend on a concept of validity. Instead of evaluating assertions or propositions separately, it takes them into consideration only as contributions to an inferential context. In this perspective, the idea of evaluating propositions separately is not appropriate. Concepts of propositional validity, and, in particular, a concept of truth, could not matter here—unless they were introduced with explicit reference to a concept of material or formal inference. The subsequent defense of connexive logic opposes the common assumption that assertions are substantial units whose meaning and value are assessed separately so that their inductive definition allows for an accurate account of concepts of logical implication. At present, assertions matter primarily as premises or conclusions. Their semantical properties are to be specified with reference to inferential contexts in which they may occur. Truth conditions and inference rules *equally* specify in what way an assertion is meaningful. The present contribution is but a preliminary sketch of a theory of connectivity. If its fallibilistic justification is developed to the full, especially as regards an adequate formal semantics, then it will certainly include a comprehensive theory of predication.

2.3. An Informal Justification of LKA

The following considerations assess inductively which well-formed sequents articulate classical inference figures. They presuppose that $a \rightarrow a$ is an admissible inference figure such that the identity of premise and conclusion indicates the *classical* assertibility of elementary propositions.[12]

Structural Rules. An inference figure $\Gamma \rightarrow \Theta$ whose revision is excluded will not lose its immunity if it is supplemented by a premise or by a partial conclusion. However, this pertains only to assertible propositions. The calculus' rules have to guarantee and thus are meant to show that inference figures pertain to no other than assertible information. Hence, the classical rules for *thinning* are admissible for elementary propositions.

The succession of items in a sequent's antecedent or consequent has but epistemic relevance. It mirrors the rank in which an information modifies another one's value. But it does not concern the immunity of an inference figure against factual revision. Therefore, the rules for *interchange* of a classical sequent calculus are admissible.

Immunity is preserved if an item listed repeatedly occurs but once. For at most, the information at issue (though not its repeated articulation) could condition any revision. Hence, the *contraction* rules of a classical sequent calculus are likewise admissible.

[12] The actual rules of LKA are listed in section 2.4.

With these structural rules all those variations of an inference figure are discussed that touch upon an assertion's role as a premise or as a partial conclusion without modifying the propositional information involved.

Rules for Logical Constants. An inference's formal validity will be preserved if, and only if, one of its premises is replaced by a particular kind of partial conclusion: In this case an information asserted by a premise is subjected to a logical operation called propositional *negation.* It compensates for the relevant kind of replacement. In order to indicate a negation of this sort, '$\neg$' will precede the relevant propositional scheme. Conversely, it is admissible to replace a partial conclusion by a premise derived in exactly the same way. In each case the resulting assertion denies what was originally asserted or denied. The resulting premise excludes what originally a partial conclusion concedes. Conversely, the resulting, partial conclusion concedes the opposite of what the former premise assumed. The negation rules of a classical sequent calculus articulate the admissibility of this interchange.

Classical sequents articulate the fallibilistic refinement of an inference figure with the help of a comma. These forms of differentiation can translate themselves into the relevant premises and partial conclusions. In this case logically complex pieces of assertible information will replace the original units. In each of them the previously asserted information is retained. But now it is subjected to a logical operation that compensates for the relevant kind of replacement. Again, logical constants indicate operations of this kind. However, the introduction of many-place constants reduces the number of assertions participating in an inference figure.

In the simplest case, two assertions contribute to an inference in the same position, i.e. as premises or as partial conclusions. It is logically admissible to articulate their joint, constructive or concessive contribution to a logical structure within one single assertion. For this purpose the pieces of information previously asserted on their own are now subjected to a logical operation. In the first case the logical relationship that allows for their integration into one single assertion is a *conjunction.* In order to indicate this relationship, '$\wedge$' is inserted between the relevant propositional schemes. In the second case the relationship at issue is a *disjunction* indicated by the insertion of '$\vee$'. Because of the system's rules for interchange as well as its rules for negation, no other forms of *unilateral connection* are taken into consideration. Two-place connectives suffice to determine the candidates to be connected and the position in which the resulting unit contributes to the inference at issue.

Likewise the logical connection between a premise and a partial conclusion can translate itself into an assertion. For the same reasons as in the previous case it does not matter whether the resulting unit is a premise or a partial conclusion. And again, the order of its components is irrelevant. Accordingly, just one form of *bilateral connection* has to be taken into account. In this case the integrating logical relationship is a *subjunction* or *material implication*. In order to indicate it, '$\supset$' is inserted between the relevant propositional schemes.

The present, fallibilistic justification of LKA obviously results in a version of classical propositional logic. But although a single connective suffices to present this kind of logic, the present approach accounts for several. Each of them stands for a logical feature of the variety of sequent in which it occurs. The preceding, epistemic presentation of logical sequents thus offers a rationale for a *particular* set of logical constants.

In the case of bilateral connection, subjunction is given preference over all other variants. Like negation, conjunction and disjunction, subjunction indicates a constitutive feature of sequents whose order and value were set forth in fallibilistic terms. The case of subjunction is particularly important: For in contrast to all other connectives, this one accounts for inference itself, and not merely for some of its features. If $\Gamma, A \rightarrow B, \Theta$ is immune to revision, so is $\Gamma \rightarrow A \supset B, \Theta$. For it says that while assuming Γ and conceding Θ, it logically follows that A conditions B. This, however, is valid if, and only if, under the very circumstances B follows from $A : \Gamma, A \rightarrow B, \Theta$.

Until now, the inferential relationship between two assertions has been discussed under the condition that they both occur in the *same* inferential context. However, the rules justified so far offer an incomplete account of how assertions relate to one another under the same logical circumstances. For within the same inferential context two assertions can occur not just *jointly*, but likewise *separately*. While the first case has been investigated, the second one remains to be discussed. Again, only the following constellations matter: Either two assertions contribute to an inference rule both as premises or as partial conclusions, or each of them matters in a different position.

If both $\Gamma, A \rightarrow \Theta$ and $\Gamma, B \rightarrow \Theta$ are logical inference figures, so is $\Gamma, A \vee B \rightarrow \Theta$. For as A and B occur separately in exactly the same context they are equivalent contributions to the relevant kind of logical inferences. These pieces of information allow not just for their joint assumption, but may also figure as components of a logically complex premise indicating that each of them by virtue of the other counts as a conceded alternative. Therefore it is admissible to infer $\Gamma, A \vee B \rightarrow \Theta$ from $\Gamma, A \rightarrow \Theta$ and $\Gamma, B \rightarrow \Theta$. The dual argumentation accounts for a conjunction in the succedent.

If both $\Gamma \rightarrow A, \Theta$ and $\Gamma, B \rightarrow \Theta$ are logical inference figures, so is $\Gamma, A \supset B \rightarrow \Theta$. An assertion of $A \supset B$ is valid if, and only if, B follows from A. Hence $\Gamma, A \supset B \rightarrow \Theta$ is immune to revision if, and only if, it is so under all circumstances that are compatible with the assumption that B follows from A. Now, to assume $A \Rightarrow B$ does not presuppose that A is the case. An inference's validity does not require that at least some or even all conditions for its application are fulfilled. Thus $\Gamma, A \supset B \rightarrow \Theta$ is valid even if instead of $A \supset B$ $\neg A$ is assumed. This assumption, however, is not justified unless it is valid to concede the partial consequence that $A : \Gamma \rightarrow A, \Theta$. If, on the other hand, $A \Rightarrow B$ does apply, then the assumption of $A \supset B$ will not hold good unless the assumption of B may replace it: $\Gamma, B \rightarrow \Theta$. Therefore it is admissible to infer $\Gamma, A \supset B \rightarrow \Theta$ from $\Gamma \rightarrow A, \Theta$ and $\Gamma, B \rightarrow \Theta$.

The first three rules for logical constants account for connections *within* an inference figure. The second trio allows the merger of two inference figures into a single one. Because of the rules for negation and interchange, it is again superfluous to consider the introduction of other two- or many-place connectives.

The rules of LKA account for all possible rearrangements of propositional inference figures that preserve their immunity against revision. For the sake of brevity a proof for the redundancy of $n + 2$ place connectives is omitted.[13]

The present, fallibilistic understanding of an inference figure justifies the rules of LKA as a whole. The epistemic sense of an inference rule motivates the choice of negation, disjunction, conjunction and subjunction as propositional connectives. The logical redundancy their introduction implies is acceptable. For each of them stands for a specific aspect of the guiding concept of adaptable information processing. The legitimacy of their introduction does not depend on the truth or falsity of the assertions involved. It results from an analysis of the preconditions under which inference figures are immune against revision.

2.4. The Calculus LKA

The subsequent presentation of LKA uses standard vocabulary and syntax of formal logic.[14]

[13]For an according proof in Dialogic Logic cf. Lorenz 1968.
[14]Cf. Section 2.1, Gentzen 1934–1935, and Zeman 1973.

LKA 1 $a \to a$

LKA 2 Interchange

LKA 2.1 $\dfrac{\Gamma, A, B, \Delta \to \Theta}{\Gamma, B, A, \Delta \to \Theta}$ LKA 2.2 $\dfrac{\Gamma \to \Delta, A, B, \Theta}{\Gamma \to \Delta, B, A, \Theta}$

LKA 3 Contraction

LKA 3.1 $\dfrac{\Gamma, A, A \to \Theta}{\Gamma, A \to \Theta}$ LKA 3.2 $\dfrac{\Gamma \to \Theta, A, A}{\Gamma \to \Theta, A}$

LKA 4 Thinning

LKA 4.1 $\dfrac{\gamma \to \theta}{\gamma, a \to \theta}$ LKA 4.2 $\dfrac{\gamma \to \theta}{\gamma \to \theta, a}$

LKA 5 Negation

LKA 5.1 $\dfrac{\Gamma, A \to \Theta}{\Gamma \to \Theta, \neg A}$ LKA 5.2 $\dfrac{\Gamma \to \Theta, A}{\Gamma, \neg A \to \Theta}$

LKA 6 Unilateral Connection

LKA 6.1 Conjunction $\dfrac{\Gamma, A, B \to \Theta}{\Gamma, A \wedge B \to \Theta}$ LKA 6.2 Disjunction $\dfrac{\Gamma \to \Theta, A, B}{\Gamma \to \Theta, A \vee B}$

LKA 7 Unilateral Fusion

LKA 7.1 Conjunction

LKA 7.1.1 $\dfrac{\Gamma \to \Theta, A \quad \Gamma \to \Theta, B}{\Gamma \to \Theta, A \wedge B}$ LKA 7.1.2 $\dfrac{\Gamma \to \Theta, B \quad \Gamma \to \Theta, A}{\Gamma \to \Theta, A \wedge B}$

LKA 7.2 Disjunction

LKA 7.2.1 $\dfrac{\Gamma, A \to \Theta \quad \Gamma, B \to \Theta}{\Gamma, A \vee B \to \Theta}$ LKA 7.2.2 $\dfrac{\Gamma, B \to \Theta \quad \Gamma, A \to \Theta}{\Gamma, A \vee B \to \Theta}$

LKA 8 Bilateral Connection, Implication $\dfrac{\Gamma, A \to \Theta, B}{\Gamma \to \Theta, A \supset B}$

LKA 9 Bilateral Fusion, Implication $\dfrac{\Gamma, B \to \Theta \quad \Gamma \to \Theta, A}{\Gamma, A \supset B \to \Theta}$

The thinning rules of LKA insert elementary assertions into inference figures that pertain solely to elementary assertions. In order to indicate a possibly empty place for a succession of terms for elementary propositions, the letters 'γ' and 'θ' are used instead of 'Γ' and 'Θ'. This rendering of LKA ensures that elementary thinning can occur only at the beginning of a derivation. Any thinning by a complex proposition has to ensue from applications of elementary thinning.

In contrast with LK, the introduction of conjunction in the antecedent or of disjunction in the succedent does not imply any thinning. The relevant propositions must be indicated *separately* before the connective is introduced. Implicit thinning will occur only in so far as a connection *reduces* the number of a sequent's formulae. In

(12) $a \rightarrow a, \neg a$

it is undecided as to whether a in the antecedent or $\neg a$ in the succedent is irrelevant. However,

(13) $a \rightarrow a \vee \neg a$

definitely contains a thinning element in the antecedent. Accordingly, the design of rules LKA 7 and 9 that merge inference figures reckon with the relevance of thinning. Whereas the corresponding rules of LK allow for it implicitly, these rules presuppose that their premises eventually have been thinned. Identical formulae in these sequents will not lead to implicit thinning either. LKA 7 and 9 prescribe their immediate contraction.

S 1 A sequent ϕ not containing any quantifiers is derivable in LK if, and only if, it is derivable in LKA.

The proof for S 1 is a simple proof by induction. It is omitted.

2.5. Logical Intentionality

Not all sequents derivable in LKA may count as formal renderings of inference rules. In each case immunity against revision is guaranteed. However, not every logically complex information preserves the absolute assertibility of its elementary components. In a number of cases the rules of LKA allow for the derivation of an inference figure that lists a complex piece of information A as being assertible, although it is not. With $A \rightarrow A$ either $A \rightarrow$ and *a fortiori* $A \rightarrow \Theta$, or $\rightarrow A$ and $\Gamma \rightarrow A$ are then derivable.

Assertibility and immunity to revision are necessary though not sufficient conditions for a complex rule of inference to be concise: Not all assertible information that LKA lets occur in the context of an inference figure actually contributes to this order. Although $A, A \supset B, C \rightarrow B$ indicates an assertion of C, this premise is irrelevant for the inference figure in which it occurs. The rules of LKA capture a concept of logical implication and subjunction that does not respect this differentiation.

Again, a still too liberal presentation of thinning permits such a broad account of implication. For this reason, sequents such as $a, b \rightarrow a \vee b$ or $a \rightarrow b \supset a$ and $a \rightarrow a \supset (b \supset a)$ seem to be paradoxical inference figures. The classical, i.e. truth-functional, concept of implication merely accounts for immunity to revision. But it does not help to articulate an order in which the genuine components of a logical inference rule are distinguished from irrelevant assertions.

Hence, a classical sequent derivable in LKA may not count as a perspicuous articulation of a form of logical orientation. Its premises, partial conclusions, and foremost, the sequent itself may contain elements that do not contribute to a conditional or inferential order, respectively. In this way, the present fallibilistic justification of classical logic leads to a critique of any understanding of logic that conceives of its semantic and syntactic aspects independently of its pragmatic sense. And, as constructive logics merely relies on a limited, i.e. purely constructive, concept of revision, the present criticism equally affects it. In contrast with these logics, according connexive systems will present themselves as *strictly inferential* varieties of logic. Not just immunity to revision justifies their set-up. Their design equally guarantees that expressions for assertible information indicate premises or partial conclusions. Moreover, it ensures that a rule of inference is always distinguishable from a context of assertions compatible with its premises or partial conclusions. Accordingly, the connexive variety of subjunction will present itself as a logical connective that reproduces an inferential order in terms of a logical relationship between an assertion's propositional components.

A connexive implication[15] $A \sqsupset B$ holds if, and only if, it is justified to *infer* B from A. Either: B follows from A for material reasons; then it is formally justified to infer B from A under the precondition $A \sqsupset B$. Accordingly, a refined version of the following sequent will be a theorem of connexive logic:

(14) $\quad A, A \sqsupset B \rightarrow B$

Or: B follows from A for purely formal reasons. Then there is no material condition for the relevant passage:

[15]Subsequently '$\sqsupset$' instead of '$\supset$' will be used in order to account for connexive implication.

(15) $A \wedge (A \sqsupset B) \rightarrow B$

(16) $\rightarrow (A \wedge (A \sqsupset B)) \sqsupset B$

If, however, $\rightarrow \neg(A \wedge B)$, an implication $A \sqsupset B$ will be neither materially nor formally valid. In this case it is by no means allowable to infer B from A.

If implication is not taken to be a mere *truth-functional* term, but is conceived in terms of *inference*, then in some sense none of the sequents (1) to (4) represents a valid rule of inference. The set-up of LKS will explain concisely in what respect these sequents do not articulate an inferential order—unless one takes them to articulate a logical consequent in a truth-functional sense.

In the remaining sections the intended, inferential concept of implication is set out. In order to identify it appropriately, a formal method will have to specify for each derivable sequent its actually irrelevant formulae. Thus, LKA will be transformed into a system LKS meeting the following two requirements. First, it will fulfil the indicated, *necessary* condition for the set-up of the connexive system—in according sequents, only expressions for assertible information indicate premises or partial conclusions. Second, it will single out those sequents standing for classical, logical consequences, though not for inference rules. Then, on the basis of LKS a connexive system LKK can present itself. In contrast with LKS it will account for the intended, inferential concept of implication not just *via negationis*, but in terms of additional rules introducing a connexive variety of subjunction.

But before this task is carried out, the question should be answered why connexive concepts of implication matter at all. Inference rules are forms of rational orientation. Those who follow them try to reach a logical goal. Accordingly, proofs are meant to present assertions as conclusions drawn from a number of premises. In this respect the inferential concept of implication accounts for an intention's basic form. For it specifies the logical conditions under which an intention's *fulfilment* is possible. The relationship between a series of premises and a conclusion or a series of partial conclusions exhibits the elementary form of all conditioned, and thus adaptable, orientation. Communicable intentions relate to their presuppositions like conclusions to their premises.

The conditions under which referring to an entity can be successful must be consistent. In particular, the identification of the relevant object may not lack consistency. The inferential forms of adaptable orientation account for this formal kind of presupposition: inconsistent premises do not allow for any conclusion. Hence, if in

LKS and LKK $A_1, \ldots, A_n \rightarrow$ is derivable, then $\Gamma, A_1, \ldots, A_n \rightarrow \Theta$ is not. In LKK $A_1, \ldots, A_n \rightarrow B$ will even allow for the derivation of $\rightarrow \neg(\neg(A_1 \wedge (\ldots \wedge A_n \ldots)) \sqsupset B)$.—Second, an intention cannot contradict the conditions under which it is supposed to be valid. It is excluded that a conclusion B contradicts its *premises* $A_1, \ldots, A_n$: If $A_1, \ldots, A_n \rightarrow B$ is derivable in LKK, so is $\rightarrow \neg((A_1 \wedge (\ldots \wedge A_n \ldots)) \sqsupset \neg B)$.—Third, the conditions of one intention cannot likewise condition yet another intention contradicting the first one: If B follows from $A_1, \ldots, A_n$ then it is logically impossible to infer $\neg B$ too. Hence

(17) $\quad (A_1 \wedge \ldots \wedge A_n) \sqsupset B \rightarrow \neg((A_1 \wedge \ldots \wedge A_n) \sqsupset \neg B)$

will turn out to be a theorem of LKK. Conditioned orientation is an elementary *form* of subjectivity. Its concept is a logical account of the order Husserl has described in terms of intentionality.[16] Phenomenology identifies this structure as a feature of consciousness. Connexive logic, however, spells out its formal properties.

3. Logical Sensibility

LKA allows identification of the problem of understanding classical propositional logic as a logic of inference. The issue translates itself into the question as to whether there can be a *pragmatic* justification of thinning. Without thinning rules a logic of inference would be incomplete. Undoubtedly,

(18) $\quad a \wedge b \rightarrow a$

is to be regarded as an inference figure. But in LKA its derivation requires an application of LKA 4.1. The role of thinning in the derivation of (17) exemplifies its logical sense.

(19) $\quad a \rightarrow a$

results in

(20) $\quad a, b \rightarrow a$

in order to make (18) derivable. But this very goal did not matter in justifying LKA 4 as being admissible. In order to account for this instrumental feature, the present understanding of a sequent will be refined subsequently. It will then count as a series of formulae in which

[16] Cf., for example, Husserl 1968, V. *Logische Untersuchung*, esp. p. 378 seq.

not all items separated by commas stand for premises or partial conclusions. Some of them, and especially those inserted by thinning, will count merely as *possible components* of premises or partial conclusions. In this way the original understanding of LKA can be preserved—as long as a sequent's formulae are not required to contribute uniformly to the derivation of inference figures.

In what follows LKA is turned into a calculus LKS such that each of its rules meets the following requirements: Their application will still allow derivation of every sequent that exemplifies the classical concept of a logical consequence. However, for each formula of a derivable sequent supplementary indicators will show whether it counts as a schematic expression for a premise or for a partial conclusion or as a possible component thereof. On the one hand the rules of LKS will articulate the semantic concept of a classical logical consequence, and on the other hand it will stand for the pragmatic concept of a classical inference figure. Both these respects will present themselves distinctly, though *within* one formal system.

In a sense the subsequent set-up of LKS restricts itself to a refinement of LKA such that the intended *pragmatic* differentiation consists in *additional* information. Under the condition of this supplementary extension an informal, semantic account of the resulting system remains unproblematic. In LKS the same sequents as in LKA are derivable. However, this situation will change if LKS is extended by rules that turn pragmatic information about inference figures into semantic information to which the system's rules may in turn be applied. In this case, the original separation vanishes.

This refinement of LKA requires the design of a new set of rules: For each propositional information a sequent mentions separately, this variety of rules will allow indication of the *role* in which the relevant information contributes to the presentation of an inference rule. In accordance with their application, each scheme for a piece of propositional information will be followed by a superscript. It will consist in a series of ciphers encoding the relevant, pragmatic information: These indicators tell *how* pieces of propositional information relate to one another within an inferential context. Each of these pieces of information serves either as an assertion's content or as a potential component of some such content. In view of their role in the presentation of inference figures, these indicators will be called *signs of logical sensibility* or, briefly, *s-labels*.[17] Their formal introduction requires design of an

[17] At present s-labels are not conceived of in algebraic terms. Nevertheless, they are similar to the kind of labels D. Gabbay has introduced recently. Cf. e.g. Gabbay 1992.

according set of rules. On the basis of LKA they will assign an s-label to each formula in a sequent ϕ. With reference to the sequence's derivation, the assigned label will indicate the formula's value for the presentation of an inference rule. Thus, only the derivability of ϕ in LKA will determine the kinds of s-labels assigned to the sequent's formulae. The required analysis must consider *all* sequents derivable in LKA. Hence, the intended justification of rules for s-labels will result from a complete induction on the construction of a sequent ϕ in LKA.

3.1. The Set-up of LKS

The only *structural* possibility in LKA to modify the set of formulae of a sequent ϕ consists in the insertion of a supplementary element. Now, the inductive sense of the present analysis merely requires us to account for the case in which a primitive element is added. All thinning by a complex unit A requires a specification of the way in which A's complexity affects its relationship with all other units contained in ϕ. But this depends entirely on the set-up of A in ϕ's derivation. Now, either the inserted formula is identical with a formula of ϕ—or it is different. For instance, the insertion can result in (20). Here b is superfluous so that it does not count as a premise. It occurs in ϕ only in so far as it contributes to the introduction of a complex unit that in turn will count as a premise or as a partial conclusion.

Every formula can receive an s-label indicating whether it already stands for the proposition of a premise or partial conclusion, or whether it is assumed to contribute to their composition. If a sequential scheme directly articulates an inference rule, then each of its members will count as standing for an assertion's proposition. None of them will be listed as a mere contribution to the making of a more complex unit. Due to LKA 1 the repeated occurrence of a in (20) counts as an embedded inference figure. Subsequently, a structure of this sort will be called a logical context or, briefly, a context. Obviously, b is not part of the context to which a on the left as well as on the right side of the arrow belongs. Here the schematic letter 'b' stands for an isolated proposition. It figures in (20) merely in order to become a part of a more complex formula. Within the logical context of a derived sequence the latter formula will then count as a schematic letter for an assertion's proposition. If in a later phase of the derivation all propositions form one, and only one, logical context, a sequent will be a direct articulation of an inference rule.

In each step of a derivation the s-label of a formula will either specify the logical context to which it belongs or it will identify the isolated proposition for which it stands. Natural numbers will allow generation of labels for contexts and isolated propositions. Each time a derivation requires a new label of this kind it will be identified by the successor of the highest natural number that has previously been used for this purpose. The context of a derivation's initial sequent is always identified by 1. A label which, as part of an s-label, indicates either some logical context or a proposition's isolation is called an *identification-label* or, briefly, an *i-label*.

Furthermore, it will be required to distinguish between the different ways in which propositions contribute to the making of a logical context. For this purpose the cipher standing for an i-label will be followed by another one—while a dot separates them from one another. This second element stands for a sign indicating the way in which the relevant proposition is *connected* with all other units belonging to the same context. This second part of an s-label is called a *connectivity-label* or, briefly, a *k-label*.

The components of a primitive context as it is articulated in a derivation's initial sequent all receive the k-label 1. An isolated proposition receives the k-label 0. Specifying the pragmatic aspects of (19) and (20) thus results in the following sequents:

(21) $a^{1.1} \rightarrow a^{1.1}$

and

(22) $a^{1.1}, b^{2.0} \rightarrow a^{1.1}$

In contrast with (20),

(23) $a, a \rightarrow a$

transforms (19) into a sequence in which *two* contexts *share* a formula. Complex s-labels make it possible to express this kind of overlapping. They consists in series of elementary s-labels separated from one another by a colon.

Hence, (23) is rendered

(24) $a^{1.1}, a^{2.1} \rightarrow a^{1.1;2.1}$

Here again a sequent contains a formula for a proposition that merely *contributes* to the indication of a premise or partial conclusion. However, the s-label 2.1 shows that in (24) the proposition a has a different inferential value than b in (22). For a with the s-label 2.1 is a *contextual alternative* of a with the s-label 1.1.

The discussion of these examples covers the pragmatic sense in which eventually repeated thinning by primitive formulae contributes to the set-up of sequents articulating inference rules.[18]

For all further rules of LKA the following question must receive an exhaustive answer: How will the present distinction between isolated propositions and eventually overlapping contexts affect a rule's application?

For the sake of brevity, the present article cannot justify all further rules of LKS in every detail. A thorough discussion of each rule, enriched by numerous examples, must give way to a concise inspection of the principles justifying the set-up of LKS.[19]

Each rule of LKA will turn as follows into a rule of LKS: Each formula occurring in a rule's premise will obtain an eventually complex s-label. Second, with reference to these assignments each formula occurring in a rule's conclusion will receive an indication for a label such that the system's rules for s-labels allow for its calculation. In this way a calculus for s-labels will be superimposed on LKA. Most of its rules specify how indications for s-labels are to be replaced by concrete formulae.

The outlined discussion on the basis of LKA will have to take into consideration the following cases: A rule either concerns a change of places or a modification of a single proposition, a connection within a single sequent or a fusion of propositions from different sequents. In the last cases it matters considerably as to whether an implication is being introduced or not. In the one case *unilateral*, in the other *bilateral* compositions are at issue. In each of these cases it has to be discussed exhaustively whether the relevant propositions are isolated or belong to one or several contexts.

—An interchange of propositions does not affect their logical contexts. The same holds for a proposition's negation in the opposite section of a sequent. The according rules of LKA are thus part of LKS; only now s-labels are assigned to all formulae the sequents contain.[20]

—A connection or a fusion of propositions pertains to the s-label of the resulting unit. For the introduced logical constant decides to which contexts the assertion of the resulting proposition belongs. In general the following principles for the calculation of s-labels hold: The *order* in which an s-label indicates overlapping contexts is irrelevant.[21]

[18]LKS 1 and LKS 5 result from this modification of LKA's initial rule and its thinning rules. As repeated application of LKS 5 on an initial sequent is possible the rule's formulation reckons with the case that elementary assertions already have s-labels. Cf. p. 30 seq.

[19]For a detailed introduction of LKS cf. Astroh 1999.

[20]Cf. LKS 3.

[21]Cf. LKS 2.1.1.

Moreover, all contexts that fully contain another one are irrelevant.[22] If they result from a contraction, a connection[23] or a fusion they must be cancelled.[24] However, propositions without context are isolated. Hence, they must be labelled accordingly.[25]

—The s-labels of a connected proposition are simply put together. However, the s-labels of fused propositions must be replaced by s-labels such that all overlapping contexts of each proposition mutually form new contexts.[26]

—Isolated propositions never allow for new contexts. Their fusion with other propositions always leads to an isolated fusion so that all involved contexts will be cancelled.[27] Unilateral connections, however, are not affected by isolated propositions. Their s-labels do not figure in the resulting unit.[28]

—Bilateral connections preserve the propositional order of contexts in so far as they reproduce an inference rule. Unless the relevant propositions are the *only* components of a context whose k-label is 1, their connection will therefore require cancellation of the involved contexts.[29]

—s-labels for propositions that eventually belong to several contexts comply with the following s-rules: contraction and unilateral connection are context preserving operations only if they do not produce contexts fully contained in others. Therefore the following regulations hold: elementary s-labels with different i-labels are put together. Elementary s-labels with identical i-labels are listed only once, and the according k-label is 1.[30]

—Fusion produces contexts. For every pair of contexts belonging to fused propositions, a fusion gives rise to a new context so that the original i-labels are replaced by new ones, whereas the original k-labels are preserved. If a proposition results from a unilateral fusion,[31] each of its elementary s-labels contains the k-label 1. If the proposition results from a bilateral fusion,[32] the k-label is 2. However, the inferential concept of an implication requires that a sequent resulting from a bilateral fusion may not contain a context listing an implication $A \sqsupset B$ in the antecedent, although $A \wedge B^{\mathsf{n.1}} \rightarrow$ is derivable in LKS.[33] Hence,

[22]Cf. LKS 2.3.2, LKS 2.3.3.

[23]What holds for unilateral connection equally holds for contraction.

[24]Cf. LKS 0.

[25]Cf. LKS 2.2.

[26]Cf. LKS 2.3.1, LKS 2.1.2; furthermore LKS 7.0.1.1, LKS 8.0.1, LKS 10.0.1.1.

[27]Cf. LKS 8.0.2, LKS 10.0.2.

[28]Cf. LKS 7.0.2.

[29]Cf. LKS 9.0.1–LKS 9.0.3.

[30]Cf. LKS 2.1.2, LKS 7.0.1.2.

[31]Cf. LKS 8.0.1, LKS 10.0.1.2.

[32]Cf. LKS 10.0.1.1.

[33]For instance, the sequents $a \supset \neg a \rightarrow \neg a$ and $\neg a \supset a \rightarrow a$ are derivable in

two contexts—each of them containing precisely one of the relevant propositions—will allow for the formation of a new context only if they are not controversial in the following sense: Two contexts m and n are *controversial* if, and only if, they belong to different sequents and differ in only one proposition belonging to different sections.[34]

4. The Calculus LKS

In addition to familiar means for the presentation of a sequent calculus, LKS will contain in particular the following units:[35]

The letters 'a', 'b', 'c' or 'd', possibly followed by a numerical subscript, will count as schematic expressions for elementary s-labels. A pair of ciphers for natural numbers separated by a dot will serve as a definite expression for an elementary s-label.

The expressions 'h', 'j', 'k', 'm', 'n' and 'o', possibly followed by a numerical subscript, will count as schematic expressions for these ciphers.

When it comes to the identification of a logical context, '$\dot{n}$' will indicate the usage of an expression for the successor of the highest natural number that has served this purpose in the preceding steps of the derivation.

The letters 'A', 'B', 'C' or 'D' , possibly followed by a numerical subscript, will count as schematic expressions for possibly complex s-labels.

Elementary s-labels are s-labels.

If A is an s-label so is A; a.

If A and B are the s-labels of two propositions A and B, then there have to be expressions for s-labels generated by logical operations on A and B: In the case of a contraction or a unilateral connection the resulting s-label is $\mathsf{A} \boxdot \mathsf{B}$. In the case of a bilateral connection it is $\mathsf{A} \odot \mathsf{B}$. In the case of a unilateral fusion it is $\mathsf{A} \boxed{\boxdot} \mathsf{B}$. In the case of a bilateral fusion it is $\mathsf{A} \circledcirc \mathsf{B}$.

If A and B are s-labels, then $\mathsf{A} * \mathsf{B}$ is an s-label resulting from a contraction, a connection or a fusion of the according propositions.

If a horizontal line is put above an expression of this form, the resulting unit will stand for that part of the indicated series of signs

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LKA. However, they do not represent inference figures. An inferential account of implication precludes acceptance of $a \supset \neg a$ or $\neg a \supset a$ as premises.

[34] Cf. LKS 10.0.1.2.

[35] In the present context formal means of representation are introduced in order to talk about certain calculi or the results of their application. Hence quotation marks come forth only if the relevant expressions are mentioned and thus not used themselves.

which is not listed in the pointed brackets. For instance, the expression '$\phi\{\mathsf{n.m}\}$' is a schematic presentation of a sequent ϕ which in at least one place contains the s-label $\mathsf{n.m}$. Consequently, the expression '$\overline{\phi\{\mathsf{n.m}\}}$' is a schematic presentation of the part of ϕ that does not contain $\mathsf{n.m}$.

If A is an s-label and ϕ is a sequent, then $\mathsf{A} \in \phi$ and $\mathsf{A} \notin \phi$ are assertions about A and ϕ. $\mathsf{A} \in \phi$ says that A occurs in ϕ. $\mathsf{A} \notin \phi$ says that A does not occur in ϕ.

An expression standing for a context's i-label will also be used in order to refer to the context itself. As few kinds of expressions are necessary in order to formulate the s-rules of LKS this ambivalence will cause no harm.

If n and m are logical contexts, then $\mathsf{n} \rightarrow \mathsf{m}$, $\mathsf{n} \leftrightarrow \mathsf{m}$, $\mathsf{n} \gg \mathsf{m}$, $\mathsf{n} \not\downarrow \mathsf{m}$, $\mathsf{n} \downarrow \mathsf{m}$, $\mathsf{n} \parallel \mathsf{m}$, $\mathsf{n} \nparallel \mathsf{m}$, $\mathsf{n} \mid 2$ and $\mathsf{n} \nmid 2$ are decidable assertions about them:

$\mathsf{n} \rightarrow \mathsf{m}$ says that due to a connection or a fusion, n is contained entirely in m.

$\mathsf{n} \leftrightarrow \mathsf{m}$ says that due to a connection or a fusion, n and m are contained in one another.

$\mathsf{n} \gg \mathsf{m}$ says that in the course of the actual derivation n was labelled *after* m.

$\mathsf{n} \downarrow \mathsf{m}$ says that n and m are *controversial*.

$\mathsf{n} \not\downarrow \mathsf{m}$ says that n and m are *not* controversial.

$\mathsf{n} \parallel \mathsf{m}$ says that n and m are *coherent*.

$\mathsf{n} \nparallel \mathsf{m}$ says that n and m are *not* coherent.

$\mathsf{n} \mid 2$ says that n is due to a bilateral fusion.

$\mathsf{n} \nmid 2$ says that n is *not* due to a bilateral fusion.[36]

LKS 0 — A Gentzen-rule of LKS is applicable to a sequent only if it does not include contexts contained in others.

LKS 1 — $a^{\dot{\mathsf{n}}.1} \rightarrow a^{\dot{\mathsf{n}}.1}$

LKS 2 — Structural rules for s-labels

LKS 2.1 — Elementary s-labels

LKS 2.1.1 — Interchange

$$\phi\{\mathtt{a};\mathtt{b}\} \Rightarrow \phi\{\mathtt{b};\mathtt{a}\}$$

LKS 2.1.2 — Contraction

$$\phi\{\mathtt{a};\mathtt{a}\} \Rightarrow \phi\{\mathtt{a}\}$$

LKS 2.1.3 — Weakening

$$\phi\{\mathtt{n.2}\} \Rightarrow \phi\{\mathtt{n.1}\}$$

[36] For the purpose of the last four relations cf. section 5.2.

LKS 2.2 Labelling of isolated propositions

$\phi\{A\} \Rightarrow \phi\{A^{\mathsf{\dot{n}.0}}\}$

LKS 2.3 Contextual s-labels

LKS 2.3.1 Connection and fusion of contexts

$\phi\{(\mathtt{a}_1;\ldots;\mathtt{a}_\mathtt{n}) * (\mathtt{b}_1;\ldots;\mathtt{b}_\mathtt{m})\} \Rightarrow$
$\quad \phi\{\mathtt{a}_1 * \mathtt{b}_1;\ldots;\mathtt{a}_\mathtt{n} * \mathtt{b}_1;\ldots;\mathtt{a}_1 * \mathtt{b}_\mathtt{m};\ldots;\mathtt{a}_\mathtt{n} * \mathtt{b}_\mathtt{m}\}$

LKS 2.3.2 Cancelling of contexts

$\phi\{\mathsf{n.k}, \mathsf{m.k}\}, \mathsf{n} \rightarrow \mathsf{m} \Rightarrow \phi\{\mathsf{n.k}, \cancel{\mathsf{m.k}}\}$

LKS 2.3.3 Contraction of contexts

$\phi\{\mathsf{n.k}, \mathsf{m.k}\}, \mathsf{n} \rightleftarrows \mathsf{m}, \mathsf{n} \gg \mathsf{m} \Rightarrow \phi\{\mathsf{n.k}, \cancel{\mathsf{m.k}}\}$

LKS 3 Interchange

LKS 3.1 $\dfrac{\Gamma, A^{\mathsf{A}}, B^{\mathsf{B}}, \Delta \rightarrow \Theta}{\Gamma, B^{\mathsf{B}}, A^{\mathsf{A}}, \Delta \rightarrow \Theta}$ LKS 3.2 $\dfrac{\Gamma \rightarrow \Delta, A^{\mathsf{A}}, B^{\mathsf{B}}, \Theta}{\Gamma \rightarrow \Delta, B^{\mathsf{B}}, A^{\mathsf{A}}, \Theta}$

LKS 4 Contraction

LKS 4.0 Rules for the s-labels in a contraction[37]

LKS 4.1 $\dfrac{\Gamma, A^{\mathsf{A}}, A^{\mathsf{B}} \rightarrow \Theta}{\Gamma, A^{\mathsf{A}\boxdot\mathsf{B}} \rightarrow \Theta}$ LKS 4.2 $\dfrac{\Gamma \rightarrow \Theta, A^{\mathsf{A}}, A^{\mathsf{B}}}{\Gamma \rightarrow \Theta, A^{\mathsf{A}\boxdot\mathsf{B}}}$

LKS 5 Thinning

LKS 5.1.1 $\dfrac{\gamma \rightarrow \theta}{\gamma, a^{\mathsf{\dot{n}.0}} \rightarrow \theta}$ provided a^{A} does not occur in θ.

LKS 5.1.2 $\dfrac{\gamma \rightarrow \theta}{\gamma \rightarrow \theta, a^{\mathsf{\dot{n}.0}}}$ provided a^{A} does not occur in γ.

LKS 5.2.1 $\dfrac{\gamma \rightarrow \theta, a^{\mathsf{A}_1}, \ldots, a^{\mathsf{A}_\mathsf{n}}, \delta}{\gamma, a^{\mathsf{\dot{n}.1};\ldots;\mathsf{\dot{n}+n-1.1}} \rightarrow \theta, a^{\mathsf{A}_1\boxdot\mathsf{\dot{n}.1}}, \ldots, a^{\mathsf{A}_\mathsf{n}\boxdot\mathsf{\dot{n}+n-1.1}}, \delta}$

LKS 5.2.2 $\dfrac{\gamma, a^{\mathsf{A}_1}, \ldots, a^{\mathsf{A}_\mathsf{n}}, \delta \rightarrow \theta}{\gamma, a^{\mathsf{A}_1\boxdot\mathsf{\dot{n}.1}}, \ldots, a^{\mathsf{A}_\mathsf{n}\boxdot\mathsf{\dot{n}+n-1.1}}, \delta \rightarrow \theta, a^{\mathsf{\dot{n}.1};\ldots;\mathsf{\dot{n}+n-1.1}}}$

LKS 6 Negation

LKS 6.1 $\dfrac{\Gamma, A^{\mathsf{A}} \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg A^{\mathsf{A}}}$ LKS 6.2 $\dfrac{\Gamma \rightarrow \Theta, A^{\mathsf{A}}}{\Gamma, \neg A^{\mathsf{A}} \rightarrow \Theta}$

[37] Cf. LKS 7.0.

LKS 7 Unilateral connections

LKS 7.0 Rules for s-labels in unilateral connections

LKS 7.0.1 Contextual s-labels

LKS 7.0.1.1 $\phi\{\mathsf{n.k} \boxdot \mathsf{m.j}\} \Rightarrow \phi\{\mathsf{n.k}; \mathsf{m.j}\}$

LKS 7.0.1.2 $\phi\{\mathsf{n.k} \boxdot \mathsf{n.j}\} \Rightarrow \phi\{\mathsf{n.1}\}$

LKS 7.0.2 S-labels for isolated propositions

LKS 7.0.2.1 $\phi\{\mathsf{n.k} \boxdot \mathsf{m.0}\} \Rightarrow \phi\{\mathsf{n.k}\}$

LKS 7.0.2.2 $\phi\{\mathsf{m.0} \boxdot \mathsf{n.k}\} \Rightarrow \phi\{\mathsf{n.k}\}$

LKS 7.0.2.3 $\phi\{\mathsf{n.0} \boxdot \mathsf{m.0}\} \Rightarrow \phi\{\sout{\mathsf{n.0} \boxdot \mathsf{m.0}}\}$

LKS 7.1 Conjunction

$$\frac{\Gamma, A^{\mathsf{A}}, B^{\mathsf{B}} \to \Theta}{\Gamma, A \wedge B^{\mathsf{A} \boxdot \mathsf{B}} \to \Theta}$$

LKS 7.2 Disjunction

$$\frac{\Gamma \to \Theta, A^{\mathsf{A}}, B^{\mathsf{B}}}{\Gamma \to \Theta, A \vee B^{\mathsf{A} \boxdot \mathsf{B}}}$$

LKS 8 Unilateral fusion

LKS 8.0 Rules for s-labels in unilateral fusions

LKS 8.0.1 Contextual s-labels

$\phi\{\mathsf{n}, \mathsf{n.k} \odot \mathsf{m.j}, \mathsf{m}\} \Rightarrow \phi\{\dot{\mathsf{n}}, \dot{\mathsf{n}}.1, \dot{\mathsf{n}}\}$

LKS 8.0.2 s-labels for isolated propositions

LKS 8.0.2.1 $\phi\{\mathsf{n}, \mathsf{n.k} \odot \mathsf{m.0}\} \Rightarrow \phi\{\sout{\mathsf{n}, \mathsf{n.k} \odot \mathsf{m.0}}\}$

LKS 8.0.2.2 $\phi\{\mathsf{m.0} \odot \mathsf{n.k}, \mathsf{n}\} \Rightarrow \phi\{\sout{\mathsf{m.0} \odot \mathsf{n.k}, \mathsf{n}}\}$

LKS 8.0.2.3 $\phi\{\mathsf{m.0} \odot \mathsf{n.0}\} \Rightarrow \phi\{\sout{\mathsf{m.0} \odot \mathsf{n.0}}\}$

LKS 8.1 Conjunction

LKS 8.1.1 $$\frac{\Gamma \to \Theta, A^{\mathsf{A}} \quad \Gamma \to \Theta, B^{\mathsf{B}}}{\Gamma \to \Theta, A \wedge B^{\mathsf{A} \odot \mathsf{B}}}$$

LKS 8.1.2 $$\frac{\Gamma \to \Theta, B^{\mathsf{B}} \quad \Gamma \to \Theta, A^{\mathsf{A}}}{\Gamma \to \Theta, B^{\mathsf{A} \odot \mathsf{B}}}$$

LKS 8.2 Disjunction

LKS 8.2.1 $$\frac{\Gamma, A^{\mathsf{A}} \to \Theta \quad \Gamma, B^{\mathsf{B}} \to \Theta}{\Gamma, A \vee B^{\mathsf{A} \odot \mathsf{B}} \to \Theta}$$

LKS 8.2.2 $$\frac{\Gamma, B^{\mathsf{B}} \to \Theta \quad \Gamma, A^{\mathsf{A}} \to \Theta}{\Gamma, A \vee B^{\mathsf{A} \odot \mathsf{B}} \to \Theta}$$

LKS 9	Bilateral connection
LKS 9.0	Rules for s-labels in bilateral connections
LKS 9.0.1	$\phi\{\mathsf{n}, \mathsf{n.k} \boxdot \mathsf{m.j}, \mathsf{m}\} \Rightarrow \phi\{\cancel{\mathsf{n}}, \dot{\mathsf{n}}.0, \cancel{\mathsf{m}}\}$
LKS 9.0.2	$\phi\{\mathsf{n.1} \boxdot \mathsf{n.1}\}, \mathsf{n.1} \notin \overline{\phi\{\mathsf{n.1} \boxdot \mathsf{n.1}\}} \Rightarrow \phi\{\mathsf{n.1}\}$
LKS 9.0.3	$\phi\{\mathsf{n.2}, \mathsf{n.1} \boxdot \mathsf{n.1}\}, \mathsf{n.1} \notin \overline{\phi\{\mathsf{n.2}, \mathsf{n.1} \boxdot \mathsf{n.1}\}} \Rightarrow \phi\{\mathsf{n.1}, \mathsf{n.1}\}$
LKS 9.1	Implication

$$\frac{\Gamma, A^{\mathsf{A}} \rightarrow \Theta, B^{\mathsf{B}}}{\Gamma \rightarrow \Theta, A \sqsupset B^{\mathsf{A}\boxdot\mathsf{B}}}$$

LKS 10	Bilateral fusion
LKS 10.0	Rules for s-labels in bilateral fusions
LKS 10.0.1	Contextual s-labels
LKS 10.0.1.1	$\phi\{\mathsf{n}, \mathsf{n.k} \circledcirc \mathsf{m.j}, \mathsf{m}\}, \mathsf{n} \not\downarrow \mathsf{m} \Rightarrow \phi\{\dot{\mathsf{n}}, \dot{\mathsf{n}}.2, \dot{\mathsf{n}}\}$
LKS 10.0.1.2	$\phi\{\mathsf{n}, \mathsf{n.k} \circledcirc \mathsf{m.j}, \mathsf{m}\}, \mathsf{n} \downarrow \mathsf{m} \Rightarrow \phi\{\cancel{\mathsf{n}}, \cancel{\mathsf{n.k} \circledcirc \mathsf{m.j}}, \cancel{\mathsf{m}}\}$
LKS 10.0.2	s-labels for isolated propositions
LKS 10.0.2.1	$\phi\{\mathsf{n}, \mathsf{n.k} \circledcirc \mathsf{m.0}\} \Rightarrow \phi\{\cancel{\mathsf{n}}, \cancel{\mathsf{n.k} \circledcirc \mathsf{m.0}}\}$
LKS 10.0.2.2	$\phi\{\mathsf{m.0} \circledcirc \mathsf{n.k}, \mathsf{n}\} \Rightarrow \phi\{\cancel{\mathsf{m.0} \circledcirc \mathsf{n.k}}, \cancel{\mathsf{n}}\}$
LKS 10.0.2.3	$\phi\{\mathsf{m.0} \circledcirc \mathsf{n.0}\} \Rightarrow \phi\{\cancel{\mathsf{m.0} \circledcirc \mathsf{n.0}}\}$
LKS 10.1	Implication

$$\frac{\Gamma, B^{\mathsf{B}} \rightarrow \Theta \quad \Gamma \rightarrow \Theta, A^{\mathsf{A}}}{\Gamma, A \sqsupset B^{\mathsf{A}\circledcirc\mathsf{B}} \rightarrow \Theta}$$

Every sequent derivable in LKS is derivable in LKA if its s-labels are removed. By construction every assignment of s-labels in LKS to a sequent ϕ is entirely due to the steps of its derivation in LKA. Hence the consistency of LKS is trivial.

S 2 LKS is consistent.

5. A System of Connexive Logic

5.1. An Intentional Concept of Logical Connectivity

The rules of LKS merely allow for a refined presentation of classical propositional logic. For each of its sequents derivable in LKA this calculus spells out in what respect it can be read as an inference rule. The key concept of this systematic differentiation is an inferential concept of implication. Labels of logical sensibility are a means of setting out the inferential value of each proposition a sequent is referring to.

In particular, the use of s-labels helps to identify those inference figures that allow derivation of one inference figure from another. They set forth the preconditions for inferring an implication from a series of implications such that it may replace them. Furthermore, the rules for the introduction of a connexive implication, summed up under LKK 9, allow one to infer from a series of implications which implication *by no means* may replace them. If, for instance, $A \sqsupset B^{\mathsf{n}.1}$ and $B \sqsupset C^{\mathsf{n}.1}$ are justified implications, then neither $A \sqsupset \neg C^{n.1}$ nor $\neg A \sqsupset C^{\mathsf{n}.1}$ can be justified.

The syntax of LKS clearly distinguishes between the semantic and pragmatic aspects of a sequent. LKK, however, abandons their strict separation. Under specific conditions LKS admits of a context-preserving connection that introduces an implication in the succedent. Under the same condition LKK allows the introduction of an implication in the antecedent. However, the relevant propositions both belong to the section in which they are connected. In this way particular constellations of s-labels allow one to derive sequential presentations of semantic structures *outside* the realm of classical propositional logic. It will become obvious that these new sequents are consistent with those derivable in LKS that contain one, and only one, context, i.e. authentic presentations of classical inference rules.

What a schematic expression '$A \sqsupset C$' stands for if it occurs in a derivable sequent now depends entirely on the contribution of its components to a form of rational orientation. The concept of implication that LKK identifies in a syntactic manner is a *connexive* concept of implication. It consists in a genuine relationship between propositions, not just between their truth values. In this way the connexive concept of implication accounts for a necessary presupposition of all conditional and *a fortiori* logical orientation.

Since its first steps the present investigation has taken into consideration that logical reasoning is not practised for its own sake, but with regard to specific subjects and purposes. The fallibilistic justification of the present kind of logic stems from this presupposition. Accordingly, the intentional concept of a connexive implication results from the intentionality of adaptable information processing. All modifications of LKS leading to LKK reflect this understanding of implication.

The connexive system LKK establishes a new relationship between logical constants and s-labels informing about a sequent's logical structure. The rules for connexive implication, i.e. LKK 9 and LKK 10, replace the separation between semantic and pragmatic information by a *mutual* relationship. The merging of these two aspects of logical and, more generally, semiotic information increases with each application of LKK 9.2 and 9.3. Whereas every sequent derivable in LKS is equivalent to a theorem of classical propositional logic, this interrelationship does not hold for LKK.

Apparently, the set-up of a connexive logic requires particular semiotic means. The superimposition of rules for s-labels onto an ordinary sequent calculus has been introduced for this purpose. They allow for a systematic understanding of the way semantic and pragmatic features of a logical and, *a fortiori*, linguistic structure depend on one another. All attempts to account for the pragmatic aspects of a language in a purely descriptive manner or, conversely, to reduce its semantics to a set of norms fail to recognize their intentional interrelationship. It is irrelevant to propose and to set up a theory about the usage or meaning of a language unless it counts as a form of *directed* articulation. The labels of logical sensibility that lead to a syntactic presentation of connexive logic are but a means of articulating this basic form of intentionality.

5.2. The Calculus LKK

Rules LKK 1 to LKK 9.1 are identical with rules LKS 1 to LKS 9.1. Their names preserve the original numbering, though the system's label is changed.

In addition to a rule for bilateral connection in the succedent the following two rules allow for a connexive introduction of implication in the antecedent:

LKK 9.2 $$\frac{\Gamma, A^{\mathsf{A}}, B^{\mathsf{B}} \to \Theta}{\Gamma, A \sqsupset B^{\mathsf{A} \boxdot \mathsf{B}} \to \Theta}$$
provided LKK 9.0.2 or LKK 9.0.3 applies.

LKK 9.3 $$\frac{\Gamma \to A^{\mathsf{A}}, B^{\mathsf{B}}, \Theta}{\Gamma, A \sqsupset B^{\mathsf{A} \boxdot \mathsf{B}} \to \Theta}$$
provided LKK 9.0.2 or LKK 9.0.3 applies.

Already in LKS uni- and bilateral constants counted as logical relations of a structurally different kind. In this preliminary system, the inferential design of implication and the fallibilistic account of disjunction and conjunction remained indifferent one against the other. Their introduction established different kinds of logical contexts. Restrictions of monotony were thus confined to the system's s-rules, but did not affect its Gentzen rules. The former were meant to distinguish steadily between isolated propositions and logical contexts, and to decide upon the *inferential value* of their modification. Hence, they affected the formal set-up of a sequent, though not the material sense of an implication linking a series of premises to their partial conclusions.

In LKS, for instance, the following sequents are derivable:

(25) $A \wedge C^{\mathsf{n.1}}, A \sqsupset B^{\mathsf{n.2}} \rightarrow B^{\mathsf{n.1}}$

(26) $A \sqsupset B^{\mathsf{n.1}} \rightarrow (A \wedge C) \sqsupset B^{\mathsf{n.1}}$

(27) $A^{\mathsf{n.1}}, A \sqsupset B^{\mathsf{n.2}} \rightarrow B \vee C^{\mathsf{n.1}}$

(28) $A \sqsupset B^{\mathsf{n.1}} \rightarrow A \sqsupset (B \vee C)^{\mathsf{n.1}}$

However, it is possible that A is a sufficient condition for B, or B a necessary condition for A—though not together with C. If not just their truth or falsity, but likewise the propositions themselves determine an implication's value, those schemata lose their inferential value. Independently of their quantificational features the following two, unjustified, inferences do not conform to the above-mentioned schemata:

(29) If one is a heavy smoker one is willing to pay a high price for tobacco.
∴ If one is a heavy smoker, but has little money one is willing to pay a high price for tobacco.

(30) If one minds one's health one is doing sports.
∴ If one minds one's health one is doing sports or excessively drinking alcohol.

LKK presents a connexive concept of implication that accounts for the relevance of supplementary conditions and alternative partial consequences. Hence, the preconditions under which bilateral fusion generates logical contexts must be defined more accurately than in LKS. *A fortiori*, this improvement will lead to a more refined assessment of the preconditions under which a context is preserved. The relevant task amounts to a translation of the following consideration into the system's rules: If two sequents $\Gamma \rightarrow A^{\mathsf{A}}, \Delta$ and $\Gamma, B^{\mathsf{B}} \rightarrow \Delta$ are supposed to undergo a bilateral fusion, the result, i.e. $A \sqsupset B^{\mathsf{A} \circledcirc \mathsf{B}}$, will not participate in comprehensive contexts unless contexts coinciding in A and B, respectively, are *coherent* in the following sense: With the exception of A and B, no proposition belonging to these contexts includes connections with isolated propositions. The question as to whether two contexts are coherent is decidable in every step of a derivation.

The s-rules of LKK must be stricter than those of LKS. New contexts may not result from a bilateral fusion unless the merging contexts, e.g. m and n, are coherent: $\mathsf{n} \shortparallel \mathsf{m}$. The s-rules for unilateral connections must be restricted accordingly.[38]

[38]However, it is not prerequisite to introduce a system of connexive logic under

Instead of LKS 7.0.2.1 and LKS 7.0.2.2 the connexive system LKK therefore contains the following rules:

LKK 7.0.2.1.1	$\phi\{\mathsf{n.k} \boxdot \mathsf{m.0}\}, \mathsf{n} \nmid 2 \Rightarrow \phi\{\mathsf{n.k}\}$
LKK 7.0.2.1.2	$\phi\{\mathsf{n}, \mathsf{n.k} \boxdot \mathsf{m.0}\}, \mathsf{n} \mid 2 \Rightarrow \phi\{\cancel{\mathsf{n}}, \cancel{\mathsf{n.k} \boxdot \mathsf{m.0}}\}$
LKK 7.0.2.2.1	$\phi\{\mathsf{m.0} \boxdot \mathsf{n.k}\}, \mathsf{n} \nmid 2 \Rightarrow \phi\{\mathsf{n.k}\}$
LKK 7.0.2.2.2	$\phi\{\mathsf{m.0} \boxdot \mathsf{n.k}, \mathsf{n}\}, \mathsf{n} \mid 2 \Rightarrow \phi\{\cancel{\mathsf{m.0} \boxdot \mathsf{n.k}}, \cancel{\mathsf{n}}\}$

Instead of LKS 10 the system LKK contains:

LKK 10	Bilateral fusion
LKK 10.0	Rules for s-labels in a bilateral fusion
LKK 10.0.1	Contextual s-labels
LKK 10.0.1.1	$\phi\{\mathsf{n}, \mathsf{n.k} \circledcirc \mathsf{m.j}, \mathsf{m}\}, \mathsf{n} \not\downarrow \mathsf{m}, \mathsf{n} \parallel \mathsf{m} \Rightarrow \phi\{\dot{\mathsf{n}}, \dot{\mathsf{n}}.2, \dot{\mathsf{n}}\}$
LKK 10.0.1.2	$\phi\{\mathsf{n}, \mathsf{n.k} \circledcirc \mathsf{m.j}, \mathsf{m}\}, \mathsf{n} \downarrow \mathsf{m}, \mathsf{n} \parallel \mathsf{m} \Rightarrow \phi\{\cancel{\mathsf{n}}, \dot{\mathsf{n}}.1, \cancel{\mathsf{m}}\}$
LKK 10.0.2	S-labels for isolated propositions
LKK 10.0.2.1	$\phi\{\mathsf{n}, \mathsf{n.k} \circledcirc \mathsf{m.0}\} \Rightarrow \phi\{\cancel{\mathsf{n}}, \cancel{\mathsf{n.k} \circledcirc \mathsf{m.0}}\}$
LKK 10.0.2.2	$\phi\{\mathsf{m.0} \circledcirc \mathsf{n.k}, \mathsf{n}\} \Rightarrow \phi\{\cancel{\mathsf{m.0} \circledcirc \mathsf{n.k}}, \cancel{\mathsf{n}}\}$
LKK 10.0.2.3	$\phi\{\mathsf{m.0} \circledcirc \mathsf{n.0}\} \Rightarrow \phi\{\cancel{\mathsf{m.0} \circledcirc \mathsf{n.0}}\}$
LKK 10.0.2.4	$\phi\{\mathsf{m}, \mathsf{m.j} \circledcirc \mathsf{n.k}, \mathsf{n}\}, \mathsf{n} \nparallel \mathsf{m} \Rightarrow \phi\{\cancel{\mathsf{m}}, \cancel{\mathsf{m.j} \circledcirc \mathsf{n.k}}, \cancel{\mathsf{n}}\}$
LKK 10.1	$\dfrac{\Gamma, B^{\mathsf{B}} \rightarrow \Theta \quad \Gamma \rightarrow \Theta, A^{\mathsf{A}}}{\Gamma, A \sqsupset B^{\mathsf{A} \circledcirc \mathsf{B}} \rightarrow \Theta}$

S 3 LKK is consistent.

Each introduction of $A \sqsupset B^{\mathsf{A} \boxbox \mathsf{B}}$ by LKK 9.2 or LKK 9.3 can be replaced by an application of LKK 7.1 or LKK 7.2, respectively. Every sequent ϕ derivable in LKK can be mapped onto a sequent ψ derivable in LKS, and conversely. The distribution of s-labels in the contexts of a sequent ψ derivable in LKS informs about the form of their derivation. Hence, it is also decidable for every formula $A \wedge B^{\mathsf{A} \boxdot \mathsf{B}}$ introduced in the antecedent and for every formula $A \vee B^{\mathsf{A} \boxdot \mathsf{B}}$ introduced in the succedent of ψ as to whether it may be replaced by $A \sqsupset B^{\mathsf{A} \boxbox \mathsf{B}}$ in the antecedent.

In contrast with the systems of McCall and Angell, the present system is neither negation inconsistent nor absolutely inconsistent. Routley's and Montgomery's criticism of these systems does not apply

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the presupposition of so strong a variety of non-monotonicity. Weaker systems of connexive logic can be formulated. However, cut elimination is not provable in every case.

to the present account of connexive logic (cf. Routley and Montgomery 1968).

Among others, the following sequents are derivable in LKK:

(31) $\rightarrow \neg(A \sqsupset \neg A)^{\mathsf{n.1}}$

(32) $A \wedge B^{\mathsf{n.1}} \rightarrow A^{\mathsf{n.1}}$

(33) $A \wedge A^{\mathsf{n.1}} \rightarrow A^{\mathsf{n.1}}$

(34) $A \wedge \neg A^{\mathsf{n.1}} \rightarrow A^{\dot{\mathsf{n}}.0}$

(35) $A \vee A^{\mathsf{n.1}} \rightarrow A^{\mathsf{n.1}}$

(36) $A \wedge B^{\mathsf{n.1}} \rightarrow B \wedge A^{\mathsf{n.1}}$

(37) $\neg(A \sqsupset \neg B)^{\mathsf{m.0}} \rightarrow A \wedge B^{\mathsf{n.0}}$

(38) $A \wedge B^{\mathsf{n.1}} \rightarrow \neg(A \sqsupset \neg B)^{\mathsf{n.1}}$

(39) $A \wedge (B \wedge C)^{\mathsf{n.1}} \rightarrow (A \wedge B) \wedge C^{\mathsf{n.1}}$

(40) $(A \wedge B) \sqsupset C^{\mathsf{n.1}} \rightarrow (A \wedge \neg C) \sqsupset \neg B^{\mathsf{n.1}}$

(41) $(A \wedge B) \sqsupset C^{\mathsf{n.1}} \rightarrow \neg((A \wedge \neg C) \sqsupset B)^{\mathsf{n.1}}$

(42) $A \sqsupset B^{\mathsf{n.1}} \rightarrow \neg(A \sqsupset \neg B)^{\mathsf{n.1}}$

(43) $A \sqsupset A^{\dot{\mathsf{n}}.0} \rightarrow \neg(A \sqsupset \neg A)^{\mathsf{n.1}}$

(44) $A \sqsupset B^{\mathsf{n.1}} \rightarrow \neg(\neg B \sqsupset A)^{\mathsf{n.1}}$

(45) $A \sqsupset B^{\mathsf{n.1}} \rightarrow \neg(B \sqsupset \neg A)^{\mathsf{n.1}}$

(46) $(A \sqsupset A) \wedge (A \sqsupset A)^{\dot{\mathsf{n}}.0} \rightarrow (A \sqsupset A)^{\mathsf{n.1}}$

(47) $A \sqsupset C^{\mathsf{n.1}}, A \sqsupset B^{\mathsf{n.1}} \rightarrow \neg(B \sqsupset \neg C)^{\mathsf{n.1}}$

(48) $A \sqsupset \neg C^{\mathsf{n.1}}, A \sqsupset B^{\mathsf{n.1}} \rightarrow \neg(B \sqsupset C)^{\mathsf{n.1}}$

(49) $C \sqsupset \neg A^{\mathsf{n.1}}, A \sqsupset B^{\mathsf{n.1}} \rightarrow \neg(B \sqsupset C)^{\mathsf{n.1}}$

(50) $A \sqsupset B^{\mathsf{n.1}}, C \sqsupset \neg B^{\mathsf{fn.1}} \rightarrow \neg(A \sqsupset C)^{\mathsf{n.1}}$

(51) $(A \vee B) \sqsupset C^{^{\mathsf{n.1}}} \rightarrow \neg(\neg A \sqsupset C)^{\mathsf{n.1}}$

(52) $(A \wedge B) \sqsupset \neg C^{\mathsf{n.1}} \rightarrow \neg(A \sqsupset (B \sqsupset C))^{\mathsf{n.1}}$

(53) $((A \sqsupset A) \sqsupset B)^{\mathsf{n.1}} \rightarrow B^{\mathsf{n.1}}$

(54) $B \wedge B^{\dot{\mathsf{n}}.0} \rightarrow A \sqsupset A^{\mathsf{n.1}}$

(55) $\neg A \sqsupset \neg(B \sqsupset \neg C)^{\mathsf{n.1}} \rightarrow \neg(A \sqsupset (B \sqsupset C))^{\mathsf{n.1}}$

(56) $A \sqsupset B^{\mathsf{n.1}}, \neg A \sqsupset C^{\mathsf{n.1}} \rightarrow (\neg(C \sqsupset B) \wedge (\neg C \sqsupset B))^{\mathsf{n.1}}$

(57) $\rightarrow (\neg A \vee ((A \sqsupset A) \sqsupset A)) \vee (((A \sqsupset A) \vee (A \sqsupset A)) \sqsupset A^{\dot{\mathsf{n}}.0}$

Due to (34), (43), (54) and (57) McCall's connexive system is not contained in LKK (cf. McCall 1966, p. 351). (32), (34), (40) and (46) show to what extent LKK admits of *simplification* and *antilogism* (cf. Routley et al. 1982, p. 83 seq.).

A fragment of LKK is sufficient for a reconstruction of categorical syllogistics in terms of propositional logic. In addition to the initial rule of LKK this fragment contains only the rules for interchange, negation and implication. A reconstruction of this sort was first proposed by MacColl. With the exception of the notation, (47), (48) and (49) reproduce this rendering of darapti, felapton and bamalip (MacColl 1878, p. 181 seq.).[39]

Some simple devices allow extension of a connexive system to a logic of conditioned belief. In this case the propositional attitude at issue does not count as a modality, but as a connexive conditional. If α is a person, z a period of time, and if A and B are propositions for which $A \rightarrow A$ is derivable, then the expression '$A \overset{\alpha}{\underset{z}{\lhd}} B$' reads as follows: α takes A to be true during z, for then he takes B to be true. On the basis of LKK, for instance, the following rule of doxastic logic can be proposed:

$$\text{D} \qquad \frac{A_1 \sqsupset B_1^{\mathsf{n.1}}, \ldots, A_n \sqsupset B_n^{\mathsf{n.1}} \rightarrow A \sqsupset B^{\mathsf{n.1}}}{A_1 \overset{\alpha}{\underset{z}{\lhd}} B_1^{\mathsf{n.1}}, \ldots, A_n \overset{\alpha}{\underset{z}{\lhd}} B_n^{\mathsf{n.1}} \rightarrow A \overset{\alpha}{\underset{z}{\lhd}} B^{\mathsf{n.1}}}$$

$n \geq 1$ and the succedent may not be empty.

(31), (40) to (42), (44), (45), (47) to (52), (55) and (56) can easily count as examples for doxastic inference rules. D presupposes logical omniscience.[40]

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[39] On the concept of implication in MacColl's early writings, cf. Astroh 1993a.

[40] LKK thus makes it possible to solve the technical problems of the epistemic logic I have sketched elsewhere (Astroh 1993b).

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