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THE RUSSELL-KAPLAN PARADOX
AND OTHER MODAL PARADOXES:
A NEW SOLUTION

The article considers some paradoxes that have been found in possible worlds semantics, such as the Russell-Kaplan paradox and a paradox proposed by Forrest and Armstrong. It is proposed that the most serious of the paradoxes can be avoided if we use as the background theory of possible worlds semantics the set theory NFU or other similar non-standard set theories instead of ZF.

1. INTRODUCTION

The most successful semantics for modal logic in the narrow sense, the logic of possibility, necessity, impossibility and contingency, has been possible worlds semantics. And though various kinds of algebraic semantics (as in Bealer 1982) are emerging as noteworthy rivals to possible worlds semantics in the area of general intensional logic (such as the logic of propositional attitudes, etc.), possible worlds semantics is still important even in that more general area. However, there is one big problem in the foundations of possible world semantics: paradoxes such as the so-called Russell-Kaplan paradox, the Forrest-Armstrong paradox, etc. These paradoxes are very serious obstacles for possible worlds semantics, especially if we want to interpret possible worlds realistically and thus have not only a modal logic, but also a possible worlds ontology. Some of these paradoxes also threaten algebraic semantics for intensional logic and thus the very possibility of intensional logic generally.

I want to propose that at least most of the paradoxes can be avoided if we use as the background theory of possible worlds semantics the set theory NFU instead of ZF. Ironically, a set theory deriving from the work of Quine, the most famous opponent of modal logic, can be used to solve the greatest problem lurking in the foundations of modal logic!

In the process of solving the paradoxes, we will also acquire a lot of new information about possible worlds. Unfortunately, to those who already find the assumption of the existence of possible worlds to be contrary to common sense, this information may make it seem even more contrary to common sense than it has seemed so far. However, the first step to logical progress is often to throw some apparently common-sensical notions into the garbage can.

The paradoxes have been presented in many forms. They have also been used for various purposes. I will first present various versions of the paradoxes. I will also discuss previous attempts to solve them and try to show that though these arguments all manage to escape formal inconsistency, they cannot be considered really satisfactory. Then I will show how all of the paradoxes are avoided in NFU. After this, I will discuss the single counterargument to my proposal that I have found in the literature. Finally, I will discuss the question of whether NFU can really be claimed to be the true set theory.¹

2. VARIOUS FORMS OF THE PARADOXES

In possible worlds semantics there are mainly two kinds of truly realistic conceptions of possible worlds (I will not consider fictionalist or other non-realist interpretations of possible worlds discourse in this article). According to Lewis (1986, p. 2) possible worlds are not only ways things might have been, but they are also something like remote planets, only they are not at any spatial distance from here nor at any temporal distance from now. Possible worlds may also be viewed as maximal states of affairs or propositions (see Plantinga 1974, pp. 44, 45) or maximal *sets* of states of affairs or propositions (see Adams 1979, p. 204). Forrest (1986) has suggested a theory in which possible worlds are not primitive entities, but are analyzed as (or replaced by) structural properties, world-natures, instead of propositions. Possible worlds are concrete entities on the approach of Lewis, but abstract ones on the approaches of Adams, Plantinga and Forrest. Approaches like theirs are often called actualist, moderate realist or abstractionist theories. Some forms of the Russell-Kaplan paradox are directed especially against the second kind of conception, but some seem to threaten both views of possible worlds. Since the views of Adams, Plantinga and

¹I am grateful for many useful comments on this article made by the anonymous referees, by professor Randall Holmes and professor Gabriel Sandu, and by various participants of the philosophical logic seminar at the University of Helsinki, including Anssi Korhonen, Panu Raatikainen and many others. All remaining mistakes are naturally my fault.

Forrest seem to be less ontologically extravagant, it is especially important to overcome versions of the paradoxes directed against them.

The paradox was first presented by Russell (1903, p. 527), in his first formulation of type theory, as one of the difficulties confronting this formulation. It may have been one cause for Russell's ultimate preference for a ramified theory of types over a simple theory of types.

If m be a class of propositions, the proposition "every m is true" may or may not be itself an m . But there is a one-one relation of this proposition to m : if n be different from m , "every n is true" is not the same proposition as "every m is true". Consider now the whole class of propositions of the form "every m is true", and having the property of not being members of their respective m 's. Let this class be w , and let p be the proposition "every w is true". If p is a w , it must possess the defining property of w ; but this property demands that p should not be a w . On the other hand, if p be not a w , then p does possess the defining property of w , and therefore is a w . Thus the contradiction appears unavoidable.

Davies (1981, p. 262) presented the paradox first in the modern literature of intensional logic in the following rather different form, saying he had heard the paradox from David Kaplan and Christopher Peacocke.²

Suppose that the cardinality of the set of fully determinate counterfactual states of affairs (possible worlds) is κ . Each subset of this set determines (or, on some accounts, is) a proposition, namely the proposition which would be expressed by a sentence which was true with respect to precisely the possible worlds in that subset. There are thus 2^κ such propositions, and 2^κ is strictly greater than κ (by Cantor's theorem). Consider some man X and time t . For each proposition it is possible that X should have been thinking a thought at t whose content would be specifiable by a sentence expressing that proposition. So there is a distinct possible situation corresponding to each such proposition, and so there are at least 2^κ possible worlds. But we began by assuming that there are precisely κ possible worlds. (I am indebted here to David Kaplan

²Kaplan's own version of the paradox has since been published in Kaplan 1995. I discovered this only while finishing this article and therefore unfortunately cannot discuss Kaplan's own version in such detail as it certainly deserves. Kaplan stresses that logic should be compatible with all kinds of metaphysical theories about real possibility. According to Kaplan (1995, p. 43) logic should not rule out the possibility that there could be sentential operators Q such that they satisfy the following schema:

$$\forall p \diamond \forall q (Qp \leftrightarrow p = q)$$

Kaplan says that it is difficult to think of natural examples of such operators; however, perhaps we can say that for every proposition, it is possible that it and only it is queried. However, no model in which propositional variables range over all subsets of W can satisfy the schema.

Kaplan suggests that the problem might be solved by arranging propositions in a ramified hierarchy, just as in Russell's type theory; however, he is by no means committed to this solution.

and Christopher Peacocke). There are, of course, things which can be said in response to this apparent paradox. But it does raise a doubt about the coherence of the notion of a fully determinate counterfactual state of affairs.

Lewis (1986, pp. 104, 105) presents the paradox in the following way, which is practically identical to the form used by Davies; however, Lewis numbers the steps in the derivation of the paradox, making it easier to search for the premise responsible for the contradiction.

1. Suppose that the cardinality of the set of possible worlds is K .
2. Each subset of this set is a proposition, namely the proposition which would be expressed by a sentence which was true with respect to precisely the worlds in that subset.
3. There are 2^K such propositions, and 2^K is strictly greater than K .
4. Consider some man and time. For each proposition, it is possible that he should have been thinking a thought at that time whose content would be specifiable by a sentence expressing that proposition; and that this should have been his only thought at that time.
5. So there is a distinct possible situation corresponding to each such proposition.
6. So there are at least 2^K possible worlds, contradicting the assumption with which we began.

Lewis responds to the paradox by denying premise (4). However, this seems at first a very arbitrary and unsatisfactory reply. Lewis justifies this rejection by appealing to what he calls a broadly functionalist theory of the content of psychological states. According to Lewis there cannot be psychological states corresponding to every proposition, since there are not functional states corresponding to every proposition.

However, Lewis's theory of psychological states is not a pure realistic functional theory. According to Lewis, the functional roles of psychological states underdetermine the assignment of content. It seems to me that if one believes in this, one can no longer be said to hold a functionalist theory of content. Lewis thinks that besides principles of fit we need "principles of humanity"; however, he does not show why such principles of humanity would not allow the existence of a psychological state directed at any proposition. Lewis seems to think that these "principles of humanity" are purely conventional. In that case we could certainly avoid paradoxes by just choosing the principles suitably. However, this theory implies that it is in part a matter of purely arbitrary decision what psychological states any person has; this seems to me to be a far from satisfactory philosophy of mind.

Jubien (1988, p. 307) presents the paradox in a stronger form. He mentions (p. 322) its resemblance to the paradox given by Russell, saying that according to Ed Gettier, David Kaplan resurrected the paradox. Unlike Lewis (but more like Davies), he thinks the paradox actually shows that possible worlds semantics is the wrong foundation for intensional logic.

First, suppose P is the set of all propositions. Let Q be its power set. Then each member of Q is a set of propositions. (Of course one of them is empty.) But now it seems that we should be able to associate with each set q in Q a proposition q^* in a one-one manner. For example, for any q we might let q^* be the proposition that *Kaplan believes some member of q* . Intuitively, if r and s are different members of Q , then the propositions r^* and s^* are also different. It therefore appears that we have a one-one function from the power-set of P into P , which contradicts Cantor's theorem.

It seems to me that the form of the paradox presented by Jubien, unlike that given by Davies and Lewis, threatens any intensional logic in which there is a set of all propositions, even if the propositions are not analyzed as sets of possible worlds. The argument given by Jubien starts from P , the set of all propositions, and it does not matter to the argument how or whether this set is analyzed. For example, an algebraic semanticist like Bealer (1982, p. 50) assumes that there is a set \mathcal{D}_0 of all propositions, and it is not at all clear how Bealer could respond to the argument, since he thinks (pp. 96, 97) that the way to find a workable solution to the version of the paradoxes arising in his logic is to adapt the best resolutions of the paradoxes in first order set theory, and he seems to think these are likely to be those used in BNG or ZF.

Patrick Grim has written a book in which he tries to prove that there cannot be any totality of all truths. This would imply that there could not be any possible worlds interpreted as Adams or Stalnaker interpret them, either, since if possible worlds were maximal sets of propositions, the actual world would have to be the set of all true propositions, i.e. of all truths, and if possible worlds were maximal propositions the actual world would have to be the maximal true proposition. Grim uses this proof against three kinds of philosophical theories: possible world theories of modality, the use of omniscience in the ontological argument and Wittgenstein's idea of the world as all that is the case.

Grim has two kinds of arguments for his claim. The first consists of various versions of the Liar. I cannot deal with these arguments in this article.³ The second kind of argument, Grim's Cantorian argument

³I will only say that the question of the correct solution of the Liar is so controversial that I do not think the paradox can be safely used to support any kind of positive statements about the nature of reality. It seems to me that if Kripke's

(Grim 1991, pp. 91–93), is very similar to the Russell-Kaplan paradox's original form presented by Russell himself. It is simpler in not using the concept of propositional attitudes but, as I will show, it is not applicable to as many theories of possible worlds as Lewis's and Jubien's form of the Russell-Kaplan argument and is therefore less general than these forms of the argument.

Grim considers the power set of the set of all truths T . According to Grim,

To each element of this power set there will correspond a truth. To each set of the power set, for example, t_1 either will or will not belong as a member. In either case we will have a truth ... There will then be at least as many truths as there are elements of the power set $\wp T$. But by Cantor's power set theorem the power set of any set will be *larger* than the original. There will then be *more* truths than there are members of T ; some truths will be left out. (Grim 1991, pp. 92, 93)

Grim draws far more radical conclusions from his paradox than Jubien from his. He thinks (1991, p. 119) that the paradox shows that all quantification over all propositions leads to contradiction.

Besides the Russell-Kaplan paradox there is also a less general modal paradox or family of modal paradoxes. It was first used by Forrest and Armstrong (1984) as an argument against Lewis's theory of possible worlds. Lewis tries to answer it (1986, pp. 101–104). This paradox takes as its target Lewis's Principle of Recombination. Informally (Lewis 1986, p. 88) the principle states that anything can coexist with anything else. Lewis states it more formally so that according to this principle, given a class of possible individuals, there is some world which copies that class, i.e. contains non-overlapping duplicates of all the individuals in that class.⁴

This paradox may not be as important for most actualist theories of possible worlds as it is for Lewis's theory. Actualists like Plantinga or Adams would probably not accept the Principle of Recombination as it stands. However, versions of the Principle of Recombination can be formulated that some actualists would accept, so the paradox may not be relevant only for extreme modal realists like Lewis.

solution to the paradox is the right one, there could indeed exist a set of all propositions. Grim shows that Kripke's solution is not intuitively satisfactory; however, I do not think it is any less satisfactory than the alternatives that do not allow a set of all propositions. There are also newer solutions to the Liar that Grim does not consider, such as the revision theories of truth developed by Herzberger, Belnap and Gupta that hold no impediment for a set of all propositions.

⁴In trying to solve the paradox questions may be raised already about whether this more exact formulation of the principle truly captures the intuitive idea, and especially about whether the addition of the qualification "non-overlapping" is required by the intuitive idea.

I can only present here the bare bones of the elaborate argument of Forrest and Armstrong.

The argument rests upon two premises.

First, every possible world is distinct from every other ...

Second, given any number of possible worlds, $W_1, W_2 \dots$, there exists a possible world, having wholly distinct parts, such that one of these parts is an internally exactly resembling duplicate of W_1 (henceforward “duplicate”), another a duplicate of W_2 , and so on.

...

Given these two premises, we claim that it follows that there can be neither the aggregate, nor the set, of all possible worlds. We begin, in this and the next paragraph, by merely outlining the argument. Suppose that such an alleged aggregate, A , exists. Consider then a very big world, W_B , which stands to the worlds which make up A , in the way already described. That is, for every world, W , which is a part of A , there will exist a proper part, P , of W_B , which internally exactly resembles W . Furthermore, each P will internally exactly resemble just one world in A . (Assuming that no two worlds exactly resemble each other. If this is denied, the argument must be, but can be, reformulated.)

W_B is not a part of A . Taking “size” in its widest sense, any W is exactly the same size as some P , a P which is a proper part of W_B . These proper parts of W_B , however, are not exactly the same size as W_B . For instance, as will be shown, W_B contains more electrons than any such P . That is to say, there is no such thing as the aggregate of all possible worlds. (Forrest and Armstrong 1984, pp. 164, 165)

Forrest and Armstrong try to support the second premise of the argument by considering different theories of co-actuality and showing that on any of them the second premise must be accepted.

Lewis presents this paradox in the following form.

Start with all possible worlds. Each of them is a possible individual. Apply the unqualified principle of recombination to this class of possible individuals. Then we have one big world which contains duplicates of all our original worlds as non-overlapping parts. But we started with all the worlds; so our big world must have been one of them. Then our big world is bigger than itself; but no matter how big it is, it cannot be that. (Lewis 1986, p. 102)

According to Lewis, Forrest and Armstrong see that this conclusion requires a subsidiary argument. Therefore the following must be added:

Suppose the big world has K electrons in it; we may safely assume that K is some large infinite cardinal. Then there are $2^K - 1$ non-empty subsets of the electrons of the big world; and for every such subset, there is a world rather like the big world in which just those electrons remain and the rest have been deleted. (I take this to be a subsidiary appeal to recombination.) Call these worlds variants of the big world. (The big world itself is one of them.) There are $2^K - 1$ variants; there are non-overlapping duplicates of all these variants within the big world; each variant contains at least one electron, therefore so

does each duplicate of a variant; so we have at least $2^K - 1$ electrons in the big world; but ex hypothesi we had only K electrons in the big world; and $2^K - 1$ must exceed K ; so the big world has more electrons in it than it has. (Lewis 1986, p. 102)

Lewis concludes from this paradox that the principle of recombination must be qualified with the proviso “size and shape permitting”, so that it says that given a class of possible individuals, there is some world which copies that class, size and shape permitting. That is, the parts of a world must be able to fit together within some possible size and shape of spacetime; however, Lewis does not think he can know what the size of a possible spacetime might be.

Though directed against the theory of Lewis, it seems to me the paradox poses a problem also for the theory of Forrest himself. In the theory of Forrest (1986, p. 19) every world-nature is a conjunction of non-relational properties P_1, P_2 , etc. together with an extra property, namely that of having no properties other than this extra property and P_1, P_2 , etc. It seems to me that one can separate the extra property (the totality-property) from every world-nature and then take the product of the properties so gotten, and then take the projection of this product and finally add the suitable extra property (the property of having no properties except the properties contained in the projection of the product of world-natures in question and itself) to the property so gotten.⁵ This results in a big world-nature, and the cardinality of the properties contained in this world-nature poses problems exactly similar to the problems with the cardinality of the electrons in the big world that Forrest and Armstrong discovered in the theory of Lewis.

Of course Forrest could say that the product in question does not exist. Indeed, he never even says there are infinite products of relations at all (and the product of world-natures in question would certainly be infinite if there are only products of any two relations). However, at least my intuitions about possibility say that it is logically possible that

⁵Perhaps I cannot assume the reader is familiar with Forrest’s concepts of product and projection. Forrest defines the operation of taking the product RxS of the relations R and S in the following way:

If R is an m -adic property or relation and S is an n -adic one, then RxS is the $(m+n)$ -adic relation which holds between $x_1, \dots, x_m, y_1, \dots, y_n$ just in case R holds between x_1, \dots, x_m and S holds between y_1, \dots, y_n .

and projection as follows:

Consider an n -adic relation R . Suppose a_1, \dots, a_n are related by R . Then, as a consequence, the sum $a_1 + \dots + a_n$ has a property, namely being the sum of parts related by R . I call this property the (monadic) projection of R .

there are worlds with infinitely many members, and to accommodate this intuition to Forrest's theory would involve using products of infinitely many relations. And if we accept infinite products it would be arbitrary to deny there are products of any set of relations.

However, while the argument thus poses a threat to at least one actualist theory of possible worlds, it does not appear to endanger such actualist theories as those of Adams and Plantinga at all.

Daniel Nolan has tried to show that neither the original argument of Forrest and Armstrong nor the reformulation of it by Lewis necessitates the restriction of the Principle of Recombination. Nolan formulates the following strengthened version of the principle that he thinks to be required in the formulation of the paradoxes: For any objects in any worlds, there is a world that contains any number of duplicates of all of those objects. Nolan (1997, p. 245) points out with regard to Lewis's version of the argument that

The principle appealed to, namely that *for any objects in any worlds, there exists a world that contains any number of duplicates of all of those objects* does not allow us to say that for a given subset of electrons in a world, there exists a world with *only* as many electrons as there are in the subset.

However, Nolan thinks (1997, p. 246) that a better argument can be constructed from a strengthened version of the Principle of Recombination to the conclusion that there cannot be a set of all possible objects.

The argument is as follows: suppose (for reductio) there is a set of all possible objects. This set must have a cardinality, as it is part of the definition of cardinality that all sets have it—call it C . But if it has a cardinality, then there must be a greater cardinality than it (e.g. the cardinality of its power-set). Call one such cardinality C^* . From the principle of recombination, for some object, there is a world that contains C^* duplicates of that object. So there are at least C^* objects to be found in worlds, so the set of all possible objects must have at least C^* members. But C^* is of course strictly larger than C —so the set of possible objects (with cardinality C) must be larger than itself. *Reductio*.

According to Nolan this is not a very bad conclusion, since there could still be a proper class of possible objects. The objection must of course immediately be raised that if the class of possible objects is a proper class, then analyzing properties (including properties of properties) as sets of possible objects in the way Lewis does and the way it is indeed very generally done in possible worlds semantics for higher order intensional logic is problematic. Nolan sees this objection and tries to answer it—however, his argument is very tentative and sketchy, and

it is doubtful if he succeeds. I cannot consider his argument here in detail.⁶

Jaakko Hintikka has also produced an argument similar to the previous paradoxes, but more simply expressed. Unlike Jubien, Forrest and Armstrong, he does not agree that the paradox would show the unsuitability of possible worlds semantics. Rather, assuming that possible worlds semantics is the right semantics for intensional logic, he uses his paradox to determine what kinds of intensional logic are possible. He thinks that the argument shows that there cannot be an alethic modal logic, a logic of logical possibility, but only such modal logics as the logic of epistemic possibility etc. According to Hintikka (1982, p. 95),

Allowing arbitrary high cardinalities in the domains of the alternatives to a given w_0 amounts to considering the class of all cardinalities as a set, and this leads to paradoxes.

3. NFU AND RELATED SET THEORIES

New Foundations (NF) is a strongly generalized and simplified version of the simple theory of types. NF was developed in Quine 1953. After developing NF, Quine developed (Quine 1955) a stronger set theory based on NF, which he called ML (Mathematical Logic). Jensen developed (Jensen 1968–69) a weakened version of NF which he called NFU (New Foundations with Urelemente). Randall Holmes has produced a textbook of set theory based on NFU (Holmes 1998a).

In NF the axiom schema of separation used in ZF is replaced by a stratified axiom schema of comprehension. A formula f_i of set theory is stratified iff there is such a function g from the formulas of set theory to natural numbers that for all subformulas f_j that occur in f_i , if f_j is of the form $f_k \in f_l$, then $g(f_l) = g(f_k) + 1$ and if f_j is of the form $f_k = f_l$, then $g(f_l) = g(f_k)$. Let us call a function g that fulfills this condition: the stratification assignment for the formula f_i . According to NF, the axiom schema of comprehension Axiom 1 holds for all

⁶Nolan gives two different proposals for solving the problem. In the second, Nolan proposes (1997, p. 252) redefining the concept of a proper class so that some proper classes might be members of other classes after all. This seems to be misusing the very concept of a proper class. However, Nolan might be groping for something not unlike what are called non-Cantorian classes in NF-style set theories (I will explain this concept later in the article). Nolan comes close to saying that the iterative conception of sets might have to be abandoned when set theory is used outside of pure mathematics, and might therefore accept my proposal of substituting NFU for ZF.

stratified formulas. NF in a pure form consists of (besides ordinary axioms of propositional and predicate logic) the stratified axiom schema of comprehension (Quine 1953, p. 92) and the axiom of extensionality 2 (p. 89).

AXIOM 1. R3'. *If ϕ is stratified and does not contain 'x', then $(\exists x)(y)(y \in x \equiv \phi)$ is a theorem.*

AXIOM 2. P1. $((x \subset y) \supset ((y \subset x) \supset (x = y)))$.

NFU is otherwise like NF, but the axiom of extensionality is qualified so that it is asserted to hold only of things that have elements (i.e. are sets). Thus Jensen uses the set abstraction schema 3 (in which \mathfrak{A} is stratified) that is simply Quine's Axiom 1 in Jensen's notation, and Axiom 4. Though NF as originally defined consists of these two axioms alone, the acronym "NFU" is usually used in later literature to refer to the combination of these two axioms with the Axiom of Infinity and the Axiom of Choice.

AXIOM 3. ABST. $\forall y \wedge x(x \in y \leftrightarrow \mathfrak{A})$.

AXIOM 4. EXT'. $\forall z(z \in x) \wedge \wedge z(z \in x \leftrightarrow z \in y). \rightarrow x \equiv y$.

Cantor's theorem does not hold unrestrictedly in these theories. In NF, NFU, ML and MLU there are non-Cantorian sets which are bigger than or equal to their power-sets. Sets for which Cantor's theorem does hold are called Cantorian. Sets x for which the function $\{\langle y, \{y\} \rangle : y \in x\}$ exists are called strongly Cantorian.

ML is related to NF as MKM (Mostowski-Kelley-Morse set theory) is related to ZF. A division is made in it between sets and ultimate classes. Quine (1968–1969, p. 320) also mentions the possibility of MLU, a variant of ML in which the axiom of extensionality is weakened in the same way it is weakened in NFU.

It seems to me that there is one very good reason to prefer NFU to ML and MLU. Let us consider how and indeed whether we could give a semantics for ML or MLU. It would seem to be a minimal desideratum for any theory (that is not itself a semantic theory) that can be taken ontologically seriously that a semantics could be given to it in a metatheory that is otherwise identical with the theory itself, but whose vocabulary contains, besides the expressions of the theory, also names referring to the expressions of the theory and predicates expressing semantic relations, and that also contains all instances of Tarski's T-schemas. However, a semantics of this kind cannot be given for ML or MLU. It is indeed not immediately clear whether a semantics of this kind can be given for NFU either; however, it is not as obviously impossible as for ML or MLU.

If there were ultimate classes, it would be impossible for us to speak of them as Quine thinks he does. Since ultimate classes are not ele-

ments of any set, they are neither arguments nor values of any interpretation function or assignment either. Thus no expression could refer to ultimate classes (not even variables under any assignment)—unless we interpreted the set-element relation as something else in the meta-language. However, thus reinterpreting the set-element relation would mean that we did not take the theory ontologically seriously. Thus ML is a self-defeating theory.

NFU, however, is not as such sufficiently strong to be a foundation for all of mathematics. However, with the additions of the axioms of Infinity and Choice, most of mathematics can be developed within it and Holmes (1998b) has shown that it can be expanded into far stronger set theories.⁷

There are also other set theories besides those deriving from the work of Quine in which there is a universal set, and to which the axiom schema of separation and Cantor's Theorem do not apply unrestrictedly, such as the set theory of Church (1974) and the positive or topological set theory of Skala (1974).⁸ In the theory of Church, separation is restricted to well-founded sets and the universal set is introduced by a special axiom. Forster 1992 is a book devoted to all set theories with a universal set;⁹ however, it concentrates mostly on the original NF. The Russell-Kaplan paradox and other modal paradoxes might perhaps also be solved in the theories of Church and Skala¹⁰ in a way similar to how I suggest they are solved in NF-style theories; however, I cannot explore this possibility in more detail here.

⁷It can be expanded to a theory called NFUA by adding to it the Axiom of Cantorian Sets (according to which all Cantorian sets are strongly Cantorian), to NFUB by adding to NFUA the rather complex-sounding Axiom of Small Ordinals (according to which for every formula ϕ of the language of NFU there is a set such that those of its elements that are Cantorian ordinals are exactly those Cantorian ordinals for which ϕ) and into NFUM by adding to NFUB the Axiom of Large Ordinals (which I will not even try to explain here). Solovay (1998) has shown that NFUB possesses the consistency strength of ZFC—with the additional assumption that there is a weakly compact cardinal. Holmes has shown that the consistency strength of NFUM is far greater than that of Kelley-Morse set theory.

⁸In addition, Barwise and Moss have recently (1996, pp. 307–312) proposed a set theory called SEC (strongly extensional theory of classes). SEC contains a distinction between sets and proper classes which is, however, not carried through in the usual way. SEC is based on Aczel's set theory ZFA—however, unlike Aczel's set theory, which does not diverge very much from ZF except in rejecting the Axiom of Foundation and replacing it with the Anti-Foundation Axiom, SEC contains both a class of all classes and a class of all sets.

⁹SEC is too recent to be mentioned in it.

¹⁰... and perhaps even in SEC.

4. A SOLUTION FOR THE PARADOXES

There are two reasons why the paradox of Jubien does not cause any trouble in NFU or related set theories. First and most importantly, as already stated, Cantor's theorem does not hold unrestrictedly in these theories. Thus if the power set of possible worlds is a non-Cantorian set, there can be an injection from its power set to the power set of its power set.

Secondly, however, the definition of the injection given by Jubien is not stratified (at least if the concept of belief can be taken as primitive) and therefore no such injection exists in NFU. Consider the injection proposed by Jubien. Formally, it is the following function f : $\{\langle x, y \rangle : (\exists z)(z \in x \wedge y = B_k(z))\}$. In NFU, a stratification assignment g has to associate the same natural number with both members of an ordered pair (see Hatcher 1982, p. 221). Thus if g were to be a stratification assignment, then for any n , $g(x) = n$ iff $g(y) = n$. Since $z \in x$ occurs in the definition of f , if g were to stratify the definition then it would have to be that $g(z) = n - 1$. Every stratifying function must also join the same number to the argument and the value of any function (this follows from the previous requirement about ordered pairs, since functions are defined as sets of ordered pairs). The relation of belief, however, has to be a function from propositions to propositions, or at least the belief operator must determine a function from propositions to propositions, and thus $g(B_k(z)) = g(z) = n - 1$ (if the belief operator did not determine a function from propositions to propositions, then the function f could not exist and the argument of Jubien would obviously fail anyway). Therefore, since $y = B_k(z)$ occurs in the definition of f , $g(y) = g(B_k(z))$ and thus finally $g(y) = n - 1$. This is, however, impossible, since for no function is it possible that $g(y) = n$ and $g(y) = n - 1$, since it cannot be so that $n = n - 1$. Therefore, no function g can be a stratification assignment for the definition of f , and therefore the definition is not stratified. Thus f does not have to exist according to NFU.

The case is different with regard to the function proposed by Grim. The definition of the function is indeed stratified, and thus the function exists. It must be noted (as Russell (1903, § 500 p. 528) already saw) that if propositions are individuated by necessary equivalence (as must be done if they are conceived as sets of possible worlds, but as can also be done even if they are viewed as primitive entities) the function is not an injection and thus not a mapping. After all, in this case the proposition that the function associates with each set of propositions would in all cases be either the contradictory proposition (the empty

set of worlds) or the trivial proposition (the set of all possible worlds). Thus the examples of Grim do not cause any problem to such logicians as David Lewis. David Lewis does, indeed, think (1986, p. 57) that there are not only propositions in the sense of sets of worlds, but also structured propositions, which he views as the meanings of sentences; however, it is not clear whether he has to assume that *they* form a set. The paradoxes of Grim may, however, pose a problem to Plantinga, against whom they were in fact primarily directed, if Plantinga thinks that propositions or states of affairs are individuated more finely and accepts ZF. However, if we use NFU instead of ZF, there is nothing contradictory in the result; it merely shows that the set of structured propositions is a non-Cantorian set.¹¹

As a matter of fact, even the power set of the set of possible worlds is probably a non-Cantorian set (and therefore the set of possible worlds is also a non-Cantorian set). Consider any stratified formula with a free propositional variable. It seems that to every such formula and every proposition there corresponds a proposition, the proposition stating that the first proposition satisfies the formula.

In fact, we can probably determine the cardinality of the power set of the set of possible worlds exactly. In NFU there is a largest cardinal, the cardinality of the universal set, $|V|$. It seems to me that the set of propositions corresponds to the universal set, and thus its cardinality is this greatest cardinal $|V|$. Now consider any formula with a free variable such that for any entity it is contingent upon whether the entity satisfies the formula. For example, consider the formula “David Lewis believes explicitly that x exists”. Even if it is necessary for some entity to exist, I think it is contingent upon whether any person explicitly believes it to exist. (This presupposes a semantics for explicit belief in which explicit belief is understood as a relation between particulars and structured propositions, where structured propositions are far more complex entities than ordinary propositions, i.e. sets of possible worlds.) It seems to me that there corresponds a proposition to this formula and any entity, the proposition that the entity satisfies the formula. Thus there seems to be an injection from the universal set to the set

¹¹Since the argument of Grim is not correct, we must also conclude that its theological application is not correct either. Omniscience cannot be proved to be impossible this easily. Of course, this does not prove that the ontological argument would be correct; that something has not been proved to be impossible would not imply that it is possible (and there remains Grim’s argument against omniscience based on the Liar paradox).

of propositions. On the other hand, propositions are also entities, and therefore the identity function is a mapping from the set of propositions to the universal set.

The solution given here to the Russell-Kaplan paradox could be used not only within the framework of possible worlds semantics, but also in the framework of an algebraic semantics. Indeed, since Bealer (1982, p. 259) leaves open the possibility that the correct resolution to the paradoxes would be an adaptation of that of Quine, it would be quite appropriate to join Bealer's logic of qualities and concepts to a theory of qualities like NFU (such as the theory of Cocchiarella (1986, 1987) already is).

To solve the paradox as presented by Davies and Lewis, we should deny premise (3). In NF and NFU, exponentiation of infinite cardinals cannot be defined as it is usually defined, because the definition would be unstratified. There is, however, another definition that usually works, but even for it 2^K is undefined (Forster 1992, p. 29) for any cardinal K that is greater than the cardinality of the set of the singletons of the members of the universal set.

The paradox of Hintikka is also solved easily. In NFU the class of all cardinalities is a set, and this leads to no paradoxes. Thus I conclude that there can after all be an alethic modal logic.

The case of the paradox proposed by Forrest and Armstrong is surprisingly not as simple as that of the Russell-Kaplan paradox or Hintikka's paradox. Most of the formulations of the paradox do not remain valid as such once NFU is substituted for ZF as the metatheory. For instance, the version of the Forrest-Armstrong paradox formulated and accepted by Nolan does not hold up as such, since it is not the case according to NFU that given a cardinality C there is always a greater cardinality C^* . However, it seems that some of the arguments can be reformulated so that they threaten the Principle of Recombination even after the substitution, since the relation of part to whole is type-level (I owe this observation to professor Randall Holmes). Since the relation between properties and their conjunctions and the relation between relations and their products are also type-level, the modification of the paradox that I have claimed threatens Forrest's own theory of possible worlds may not be solved by the change of set theories either. However, unlike Lewis with his worlds, Forrest does not have to hold that his world-natures are wholly distinct, and this might prevent reformulating the paradox so that it would apply to Forrest's theory. In any case, the paradox does not threaten most actualist theories of possible worlds.

5. COUNTERARGUMENTS

Most people who have presented modal paradoxes have simply assumed without argument that the right theory in whose context to consider the paradoxes is ZF. Jubien does consider the possibility of rejecting standard set theory in favor of a weaker set theory such as Kripke-Platek. However, we can also reject standard set theory in favour of NFUM, which has a greater consistency strength than ZFC.

Jubien does not seem to have any idea that the paradox he presents could be solved by adopting a set theory like NF, NFU or ML. However, another attacker of modal logic, Patrick Grim, is aware of the possibility and actually tries to argue against it. Fortunately, his arguments do not seem very convincing.

Grim (1991, p. 100) admits that NF might offer an alternative set theory in which a place could be found for a set of all truths. However, he argues that the adoption of such a theory to save a set of all truths proves to be a very desperate move. The costs of NF, he says, prove to be enormous. He mentions, for instance that the axiom of choice can be disproved in NF. Most crucial, however, in his view is the fact that induction over unstratified conditions is not possible in NF. Grim, however, is not aware that there exists a variant of NF, NFU, in which some of the costs he mentions need not be paid (NFU had been in print for over twenty years when Grim wrote his book). The axiom of choice is consistent with NFU. Induction over unstratified conditions is indeed not possible in NFU, either; however, I cannot see why such induction would be so crucial.

Since Grim thinks NF can be dismissed by mentioning its costs, he next turns to ML. However, his arguments against a set of all truths existing if a set theory based in ML is true are not very impressive.

According to Grim (1991, p. 102):

What then are the prospects for a set or class of all truths within an ML-like system? Consider first a variant on ML in which we admit truths “at the bottom”, as it were, as urelements suitable for set membership. To change ML as little as possible in the process, let us leave stratification requirements on sets of sets as they stand and continue to accept classes only of sets. We will thus not provide, at this point, for classes of truths. On the other hand, let us leave *sets* of truths unrestricted; for any condition C, there will be a set of those truths satisfying C.

It seems to me that Grim totally misrepresents what he does. He says that he wants to change ML as little as possible so that truths can be treated as urelements within it. To proceed as he does, however, is not to change ML as little as possible. It is essential to a solution of paradoxes in the style of ML as well as in that of NF that sets, of

any kind, only correspond to stratified conditions. ML adds to this the requirement that only elements of sets be quantified over in such conditions. If we admit to ML entities other than sets, then we have to impose stratification requirements on all entities that are not proper classes (as would be done in MLU). Thus what we should say about truths is that for any *stratified* condition C, in which only entities that are not ultimate (proper) classes are quantified over, there will be a set of those truths satisfying C. Curiously, Grim does not consider this kind of theory at all. He only considers theories in which there is a set of all truths satisfying predicative conditions, etc. Also, one could consider an expansion of ML in which truths were treated as sets of worlds, as Lewis does, instead of being taken as primitives; in such a theory there obviously need not always be sets of truths satisfying an unstratified condition.¹²

Thus we must conclude that Grim has not shown that a set of all truths cannot be assumed to exist in a set theory based on ML. However, since there are better theories available that Grim does not consider at all, NFU, NFUA, NFUB and NFUM, I do not think we need to use ML, especially in view of the semantic difficulties with ML and MLU I mentioned earlier.

Jubien (1988, pp. 113–122) also has an argument that even the mere quantification over all propositions leads to trouble. This argument is intended to refute Plantinga's theory of possible worlds. However, it seems to be the worst of his arguments.

But putting sets aside, cannot we speak quantificationally of a *property* shared by all and only those propositions that are in fact true?

It appears not. For consider any property T suggested as filling such a role. Without yet deciding whether T does in fact do what it is supposed to, let us call all those things to which T *does* apply *t*'s. Consider further

1. any property that in fact applies to nothing
2. all properties that apply to one or more *t*'s, to one or more of the things to which T in fact applies.

We can now show that there are strictly more properties referred to in (1) and (2) than there are *t*'s to which T applies. Suppose any mapping *f* of *t*'s one-to-one properties referred to in (1) and (2). Can any such mapping assign a *t* to every such property? No. For consider the property D:

PROPERTY D The property of being a *t* to which *f*(*t*)—the property it is mapped to by our chosen *f*—does not apply.

Thus according to Grim, propositional quantification together with a notion of properties leads to contradiction. The answer, of course,

¹²Grim also has an argument which he thinks shows that there cannot even be a *class* of all truths, even if we use ML as a metatheory. The line of reasoning in this argument does not seem very clear to me.

is that we cannot assume unrestricted comprehension for properties any more than for sets. We must restrict comprehension for properties somehow. If we accept NFU for sets, the natural procedure is to extend the definition of stratification to formulas of higher order predicate logic and say that only stratified formulas of higher order predicate logic with one free variable correspond to properties. This is actually done by Nino Cocchiarella in his HST* (see Cocchiarella 1986, 1987).

If one thinks that ZF is the true set theory, on the other hand, one would naturally use some kind of separation schema also in his theory of property existence. If Plantinga did this, I am not sure his theory would fall victim to any of the paradoxes proposed by Grim (unlike the theory of Adams, which apparently cannot be saved otherwise than by adopting an NF-style set theory).

6. A COMPARISON OF NFU AND ZF

If NFU and ZF were equally good as foundations of mathematics, then the fact that NFU offers a better foundation for the semantics of modal logic would, in my opinion, be enough to make us prefer it to ZF. However, it must be admitted that in this case the superiority of NFU would be rather precarious. However, I think that NFU is superior to ZF even considered simply as a foundation for mathematics. Thus it seems to me that we have very good reasons indeed to accept NFU instead of ZF.

It is obviously impossible to go through all the reasons why I think NFU is a better foundation for mathematics than ZF and to refute all the contrary arguments in this paper. However, I will indicate briefly the most important of my reasons.

The constructions of natural numbers, cardinals and ordinals in NFU are far more intuitive and closer to those originally used by Cantor and Frege in naive set theory, than those used in ZF. Indeed, it could be argued that the constructions used in ZF could never have been discovered by Cantor or Frege if they had conceived of the principles of ZF and believed in their truth from the beginning. The only way Zermelo and Fraenkel could develop them was to take the constructions of Cantor and Frege that were already at hand and transform them to something that exists according to ZF by hook or crook. This is done by finding what are called canonical representatives. However, it is apparently completely arbitrary what canonical representatives are chosen. Take for example the natural number two. In NFU, as in Frege's theory, it is identified with the set of all sets with two members. To dispel the appearance of circularity, this is further analyzed as the

set of all sets x such that they have members y and z such that y is not identical with z and for all u , if u is a member of x it is identical with y or with z . If numbers must be analyzed as sets, then this is a very natural analysis of numbers, in which there seems to be nothing arbitrary. The only trouble with it is the question whether numbers should be identified with sets at all, or whether they should rather be identified with properties (as, for example, Bealer (1982, pp. 120–143) proposes). In ZF, however, the number two is identified either with the set consisting of the numbers zero and one or with the set containing the number one, etc. This raises the problem discussed by Benacerraf (1983).

If the numbers constitute one particular set of sets, and not another, then there must be arguments to indicate which. . . . In awaiting enlightenment on the true identity of 3 we are not awaiting a proof of some deep theorem. Having gotten so far as we have without settling the identity of 3, we can go no further. We do not know what a proof of that *could* look like.

Benacerraf (1983, p. 292) infers from this that numbers are not objects. According to him,

There are not two kinds of things, numbers and number-words, but just one, the words themselves.

He thinks this view differs from the kind of extreme formalism that fails to assign any meaning whatsoever to the statements of number theory. However, I cannot see any difference.

According to Benacerraf, if numbers are to be identified with classes of classes, the same should be done with all quantifiers—and I certainly agree. However, also according to Benacerraf (1983, p. 284) this is impossible, since

. . . in no consistent theory is there a class of all classes with seventeen members, at least not alongside the other standard set-theoretical apparatus.

Benacerraf is wrong in this. NFU does have a class of all classes with seventeen members, and pretty much all the standard set-theoretical apparatus as well. As for what standard apparatus it does not have, good riddance to it! Indeed, the existence of a class of all classes with seventeen members is also consistent with the set theories of Church and Skala, though it does not follow from them as it follows from NFU.

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