EVENTS AND TIME IN A FINITE AND CLOSED WORLD

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There are numerous occasions on which we need to reason about a finite number of events. And we often need to consider only those events which are given or which we perceive. These give rise to the Criteria of Finiteness and Closedness. Allen's logic provides a way of reasoning about events. In this paper I examine Allen and Hayes' axiomatisation of this logic, and develop two other axiomatisations based on the work by Russell and Thomason. I shall show that these three axiomatisations are weakly equivalent, and that only the last two meet the Criteria of Finiteness and Closedness (to different degrees). I shall then examine two ways of constructing instants of time in a finite and closed world, i.e. the Russell construction and the Thomason construction. I shall prove that these two constructions are equivalent under certain conditions.

1. Introduction

Various theories have been devised in the past to explain what time is. Broadly speaking, they fall into two categories: absolute theories and relational theories (see Lin 1991 for a detailed exposition). Absolute theories take time elements, such as instants or intervals, as primitive, and regard events (occurrences, or processes, etc.) as happening in time. These theories can be further divided into two types. One type treats instants of time as primitive, as exemplified in Newton (1934) and McDermott (1982). The other type takes intervals of time to be elementary, e.g. Newton-Smith 1980, Hamblin (1972), Dowty (1979), and Allen and Hayes (1985, 1989). Relational theories of time, on the other hand, hold that our concept of time is derived (abstracted) from our experiences of events and thus regard events as more fundamental than time. Exponents of such theories include Whitehead (1920), Russell (1926, 1956), Kamp (1979, 1980), Bach (1981), Turner (1984), and Thomason (1984, 1989). In this paper I take the relational view of time, not only because it is psychologically plausible (Zward 1976, O'Connor and Hermelin 1978, Piaget 1954, Michon 1989) but also because it is practically useful (Bach 1981, Davidson 1967, Kamp 1979, 1980, Parsons 1990).

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Most formal theories of time, such as the ones presented in Lin (1991, 1994), presume an infinite number of events. But in this paper I concentrate on the construction of time from a *finite* number of events. The motivation for doing so is this. On the one hand, we often need to reason about a finite number of events; on the other hand, our concept of time is derived from the events we perceive, which are finite in number. Let us define a *world* as consisting of a set of events and their temporal relationships. We are thus interested in events and time in a finite world here. Apart from finiteness, we also want the world to be *closed*. To explain the concept of a closed world, let us consider the world depicted in figure 1.

In this world, there are only three events, e_1, e_2 and e_3, describing, say, the event of my having breakfast, the event of my driving to work, and the event of my listening to the radio. But of course there are events (in the universe) which are not included in this world. For example, there are bound to be events before e_1, events after e_2, and events between e_1 and e_2. But if the world is closed, we should not think about such events, which lie outside the present world. In other words, we should only deal with events given to us. The idea of a closed world is similar to that of *nonmonotonic reasoning* in artificial intelligence, which tells us that jumping to conclusions when faced with incomplete information is an important part of our common-sense reasoning (Gabbay *et al.* 1993).

A formal theory of time consists of (at least) two parts: an axiomatisation of the events and their temporal relationships in a world, and a method of constructing instants of time out of events. The criterion that the world should be finite and the criterion that the world should also be closed both impose restrictions on a formal theory of time (it is to be noted that the two criteria are independent of each other). They require i) that the axiomatisation and the construction should not assume that there are an infinite number of events, and ii) that they should not assume that there are events outside the world in question. Thus, if a theory of time for the world of figure 1 has the following axiom:

$$\forall e(\exists e'(e' < e))$$

then this theory will violate the assumption that the world in figure 1 is closed (because it assumes that, for instance, there is an event before e_1 in

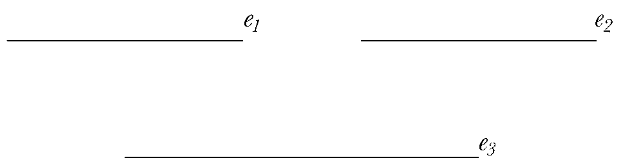

Figure 1. Events in a closed world.

that world). Furthermore, if time in this world is not circular, then the above axiom will imply that there are an infinite number of events. And this will violate the assumption that the world in figure 1 is finite.

In this paper I construct a theory of time for a finite and closed world, that is, a theory of time that satisfies the criteria of Finiteness and Closedness. The task naturally divides into two parts. First, I must axiomatise the events of a finite and closed world and the temporal relationships among the events. I examine the axiomatisation by Allen and Hayes (1985, 1989), and develop two others based on the work of Russell (1926, 1956) and Thomason (1984, 1989). As we shall see, only the latter two meet the criteria of Finiteness and Closedness (to different degrees). The second part of our task is to construct instants of time on the basis of the Russell and Thomason axiomatisations. In undertaking this task I reveal some logical relationships among the three axiomatisations, and between Russell's and Thomason's constructions of instants out of events. All theorems contained in this paper are original.

2. Logic of Events in a Finite and Closed world

2.1. Allen's Logic of Events and Allen-Hayes' Axiomatisation

There are numerous occasions on which we need to reason about a finite number of events. Given certain explicit relationships among a set of events we often need to infer additional relationships implicit in the ones given. For example, planning, causal reasoning, and story understanding, etc. What kind of relationships can stand between two events? How can we deduce the implicit relationships among a set of events from those explicitly given? Allen (1983) puts forward a logic of events which allows us to perform such deductions.[1]

According to Allen (1983) there are thirteen possible temporal relations between two events, viz: "before (*b*)", "after (*bi*)", "during (*d*)", "contains (*di*)", "overlaps (*o*)", "overlapped-by (*oi*)", "meets (:)", "met-by (:*i*)", "starts (*s*)", "started-by (*si*)", "finishes (*f*)", "finished-by (*fi*)", and "equals ($\equiv$)". If the temporal relation between two events (e_1 and e_2) is unknown, then the relation can be any one of the thirteen relations just mentioned; and we represent this situation as $e_1 \rhd\lhd e_2$. Allen devises a transitivity table (table 1, summarised from Allen 1983), using which the possible relations

[1] Allen, and also Hayes, actually used the term intervals of time (Allen 1983, 1984, Allen and Hayes 1985, 1989). Since in this paper I concentrate on the logical relationships between Allen and Hayes' theory and some other event-based theories to be discussed shortly, rather than their philosophical differences (Lin 1991), I shall regard the basic elements in their theory as events rather than intervals. In any case, Allen and Hayes' logic also applies to events (with only one slight qualification, see Note 3 below).

	b	d	f	:	o	s	≡	si	oi	:i	fi	di	bi
b	b	b d : o s	b d : o s	b	b	b	b	b	b d : o s	b d : o s	b	b	⋈
d	b	d	d	b	b d : o s	d	d	bi d f :i oi	bi d f :i oi	bi	b d : o s	⋈	bi
f	b	d	f	:	d o s	d	f	bi :i oi	bi :i oi	bi	f fi ≡	bi di :i oi si	bi
:	b	d o s	d o s	b	b	:	:	:	d o s	f fi ≡	b	b	bi di :i oi si
o	b	d o s	d o s	b	b : o	o	o	di fi o	d di f fi o oi s si ≡	di oi si	b : o	b di fi : o	bi di :i oi si
s	b	d	d	b	b : o	s	s	s si ≡	d f oi	:i	b : o	b di fi : o	bi
≡	b	d	f	:	o	s	≡	si	oi	:i	fi	di	bi
si	b di fi : o	d f oi	oi	di fi o	di fi o	s si ≡	si	si	oi	:i	di	di	bi
oi	b di fi : o	d f oi	oi	di fi o	d di f fi o oi s si ≡	d f oi	oi	bi :i oi	bi :i oi	bi	di oi si	bi di :i oi si	bi
:i	b di fi : o	d f oi	:i	s si ≡	d f oi	d f oi	:i	bi	bi	bi	:i	bi	bi
fi	b	d o s	f fi ≡	:	o	o	fi	di	di oi si	di oi si	fi	di	bi di :i oi si
di	b di fi : o	d di f fi s si o oi ≡	di oi si	di fi o	di fi o	di fi o	di	di	di oi si	di oi si	di	di	bi di :i oi si
bi	⋈	bi d f :i oi	bi	bi d f :i oi	bi d f :i oi	bi d f :i oi	bi	bi	bi	bi	bi	bi	bi

Table 1. The transitivity table for the thirteen temporal relations.

between two events can be immediately obtained if it is known that they each stand in a definite relation to a third event. For example, suppose that e_1 overlaps e_2 (i.e. $e_1\ o\ e_2$) and e_2 contains e_3 (i.e. $e_2\ di\ e_3$), then table 1 tells us that it is the case that e_1 is before, or overlaps, or meets, or contains, or is finished by e_3, that is, $e_1\ (b\ di\ fi : o)\ e_3$. In addition to this transitivity table, Allen (1983) also provides a general algorithm which, given some relations between a set of events, can compute the relationship between any pair of events in the set.

Allen's logic of events has a defect, however: it is not axiomatised. This makes it difficult to study the formal properties of the logic. For example, it is difficult to know whether the logic is sound, that is, whether all the derivations allowed by the logic are correct. To overcome this deficiency Allen and Hayes (1985) present an axiomatisation for Allen's logic of events. They (1985, 1989) discover that all temporal relations between events can be defined in terms of a single relation "meet (:)". (The intuitive meaning of one event's meeting another is that there is no time between these two events and no time that they share.)[2] For example, the relation "equal ($\equiv$)" and the relation "is_union_of ($+$)" can be defined as:

$$e \equiv e' =_{def} \exists e_1 e_2 (e_1 : e : e_2 \wedge e_1 : e' : e_2);^3$$

$$e \equiv e_1 + e_2 =_{\text{def}} \exists e_3 e_4 (e_3 : e : e_4 \wedge e_3 : e_1 : e_2 : e_4).$$

The relation "meet (:)" is assumed to satisfy the following postulates A1–A5:

A1: $\forall e_1 e_2 (\exists e_3 (e_1 : e_3 \wedge e_2 : e_3) \rightarrow \forall e_4 (e_1 : e_4 \leftrightarrow e_2 : e_4))$;
A2: $\forall e_1 e_2 (\exists e_3 (e_3 : e_1 \wedge e_3 : e_2) \rightarrow \forall e_4 (e_4 : e_1 \leftrightarrow e_4 : e_2))$;
A3: $\forall e_1 e_2 e_3 e_4 (e_1 : e_2 \wedge e_3 : e_4 \rightarrow e_1 : e_4\ xor\ \exists e_5 (e_1 : e_5 : e_4)\ xor\ \exists e_6 (e_3 : e_6 : e_2))$;[4]
A4: $\forall e_1 (\exists e_2 e_3 (e_2 : e_1 : e_3))$;
A5: $\forall e_1 e_2 (e_1 : e_2 \rightarrow \exists e_3 e_4 e_5 (e_5 \equiv e_1 + e_2 \wedge e_3 : e_1 : e_2 : e_4 \wedge e_3 : e_5 : e_4))$.

The intended meanings of these postulates are self-evident and I shall not repeat them here. Other relations, such as "before", "during", and so

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[2] Allen and Hayes distinguishes between the case where an event e_1 meets an event e_2 and the case where e_1 is before e_2. For them, the latter means that there is an event which is abutted by e_1 at one end and by e_2 at the other. Of course, if e_1 meets e_2 then e_1 is before e_2 in some sense. But this is not the sense Allen and Hayes are using. To avoid confusion, Allen and Hayes use "*b*" to express their sense of "before" (see the relevant definition below).

[3] Allen and Hayes (1985, 1989) actually regard $p \equiv q$ as saying that p and q are the same interval. Since we concentrate on events only, so by $e \equiv e'$ we mean that e and e' are temporally simultaneous. The logic is not affected by this difference.

[4] "*xor*" stands for *exclusive or*.

on, can be defined in terms of the relation "meet":

$$e\ b\ e' =_{\text{def}} \exists e_1(e : e_1 : e');$$

$$e\ o\ e' =_{\text{def}} \exists e_1e_2e_3e_4e_5(e_1 : e : e_4 : e_5 \wedge e_1 : e_2 : e' : e_5 \wedge e_2 : e_3 : e_4);$$

$$e\ s\ e' =_{\text{def}} \exists e_1e_2e_3(e_1 : e : e_2 : e_3 \wedge e_1 : e' : e_3);$$

$$e\ f\ e' =_{\text{def}} \exists e_1e_2e_3(e_1 : e_2 : e : e_3 \wedge e_1 : e' : e_3);$$

$$e\ d\ e' =_{\text{def}} \exists e_1e_2e_3e_4(e_1 : e_2 : e : e_3 : e_4 \wedge e_1 : e' : e_4);$$

$$e\ {:}i\ e' =_{\text{def}} e' : e;$$

$$e\ bi\ e' =_{\text{def}} e'\ b\ e;$$

$$e\ oi\ e' =_{\text{def}} e'\ o\ e;$$

$$e\ si\ e' =_{\text{def}} e'\ s\ e;$$

$$e\ fi\ e' =_{\text{def}} e'\ f\ e;$$

$$e\ di\ e' =_{\text{def}} e'\ d\ e;$$

$$e \bowtie e' =_{\text{def}} e\ (b\ bi\ :\ {:}i\ d\ di\ f\ fi\ o\ oi\ s\ si\ \equiv)\ e'.$$

With these definitions and A1–A5 the entire transitivity table (table 1) can be derived, that is, they constitute an axiomatisation of Allen's event logic (see Allen and Hayes 1985, 1989).

From Allen and Hayes's axiomatisation we can also easily prove the following intuitive relationships: ":" and "*b*" are irreflexive, and are mutually exclusive. Formally, this is:[5]

Theorem 1.

P1: $e_1 : e_2 \rightarrow \neg e_1\ b\ e_2$;
P2: $\neg e : e$;
P3: $e_1 : e_2 \rightarrow \neg e_2 : e_1$;
P4: $\neg e\ b\ e$;
P5: $e_1\ b\ e_2 \rightarrow \neg e_2\ b\ e_1$.

Proof. The proof makes central use of the exclusiveness in axiom A3. For example, let us prove P1. *Proof.* Suppose $e_1 : e_2$. Suppose further that $e_1\ b\ e_2$. Then according to the definition of "before (*b*)", there must be an event e_3 such that $e_1 : e_3 : e_2$. So we have $e_1 : e_2$ and $e_3 : e_2$. By A3 there should be three mutually exclusive cases: i) $e_1 : e_2$, ii) $e_1\ b\ e_2$, and iii) $e_3\ b\ e_2$. So case i) and case ii) should be mutually exclusive. But this violates our assumption that both cases hold. □ P1 can then be used to prove P2–P5.

[5] Some other intuitive relationships can also be proved. See Section 2.2 below, especially Theorem 3.

For example, let us prove P2. *Proof.* Suppose $e : e$. Then we have $e : e : e : \ldots$ So we have both $e : e$ and $e\ b\ e$. But this violates P1. □ □

However, Allen and Hayes' axiomatisation fails to satisfy the criteria of Finiteness and Closedness. For example, the definition of "before (b)" requires that there must be an event between the two events in question. But in a closed world like the one described in figure 1, it is true that $e_1\ b\ e_2$, but there is no intermediate event between the two events. So the Allen-Hayes axiomatisation may fail the Closedness Criterion easily. Apart from this, this axiomatisation violates the Finiteness Criterion. For example, A4 says that for every event there is one event meeting it and one which it meets. Take any event e_1. A4 says that there there must be an event e_2 such that $e_1 : e_2$. This e_2 cannot be identical to e_1, otherwise we would have $e_1 : e_1$ and this would violate P2. So e_2 must be a different event. Now consider e_2. According to A4, there must be an event e_3 such that $e_2 : e_3$, and hence $e_1 : e_2 : e_3$. e_3 must be different from both e_1 and e_2: if e_3 were identical to e_1, $e_1\ b\ e_1$ would be true and hence P4 would be violated; if e_3 were identical to e_2, $e_2 : e_2$ would be true and P2 would be violated. If we repeat this process, we will see that there must be an infinite number of events.

In the rest of this section I present two other axiomatisations which do satisfy the criteria of Finiteness and Closedness.

2.2. The Axiomatisation Based on the Russell Event Structure

Russell claims that our knowledge of the external world is derived from our sense data. In the case of time, he holds that instants of time, which are durationless, cannot be perceived, and he suggests that they must be constructed from events and their temporal relationships which we do perceive. In this section I discuss how Russell formalises the temporal relationships between events. In Section 3.2 below I present his way of constructing instants out of events and their temporal relationships. According to Russell (1926, 1956), the fundamental relations between events are that of before-and-after or precedence ("$<$"), and that of simultaneity or overlap ("$\bigcirc$"). There are several general properties which belong to these relations, and these relations can be justified in terms of the intended meanings of "$<$" and "$\bigcirc$" and our intuitions regarding events. Firstly, "$<$" is asymmetric but transitive. Secondly, "$\bigcirc$" is symmetric and reflexive. Thirdly, there are properties involving both "$<$" and "$\bigcirc$": if one event precedes another, it does not overlap it; if one event precedes another, and this other event overlaps a third which precedes a fourth, then the first precedes the fourth; for any two events, either they overlap or one precedes the other. In addition to "$<$" and "$\bigcirc$", events can be related in other ways. For example, an event can "meet $(:)$" another, "begins before $(<_B)$" another or "ends before $(<_E)$" another. Let us formalise all this in the definition below (taken from Kamp 1979, 1980, and slightly modified):

DEFINITION 1. *The Russell event structure* $\Sigma_R = \langle E, <, \bigcirc, :, <_B, <_E \rangle$ *consists of a non-empty set E of events together with five binary relations: "before* $(<)$*", "overlap* $(\bigcirc)$*", "meets* $(:)$*", "begins-before* $(<_B)$*", and "ends-before* $(<_E)$*". These relations are assumed to satisfy the following axioms* R1–R10:

R1: $e_1 < e_2 \rightarrow \neg e_2 < e_1$;
R2: $e_1 < e_2 \wedge e_2 < e_3 \rightarrow e_1 < e_3$;
R3: $e_1 \bigcirc e_2 \rightarrow e_2 \bigcirc e_1$;
R4: $e_1 \bigcirc e_1$;
R5: $e_1 < e_2 \rightarrow \neg e_1 \bigcirc e_2$;
R6: $e_1 < e_2 \wedge e_2 \bigcirc e_3 \wedge e_3 < e_4 \rightarrow e_1 < e_4$;
R7: $e_1 < e_2 \vee e_1 \bigcirc e_2 \vee e_2 < e_1$;
R8: $e_1 : e_2 \leftrightarrow e_1 < e_2 \wedge \neg \exists e_3 e_4 (e_3 \bigcirc e_4 \wedge e_1 < e_4 \wedge e_3 < e_2)$;
R9: $e_1 <_B e_2 \leftrightarrow \exists e_3 (e_3 \bigcirc e_1 \wedge e_3 < e_2)$;
R10: $e_1 <_E e_2 \leftrightarrow \exists e_3 (e_1 < e_3 \wedge e_3 \bigcirc e_2)$.

Let us define "$=_B$" (begins at the same time) and "$=_E$" (ends at the same time) as:

DEFINITION 2.

$$e_1 =_B e_2 =_{\text{def}} \neg e_1 <_B e_2 \wedge \neg e_2 <_B e_1;$$

$$e_1 =_E e_2 =_{\text{def}} \neg e_1 <_E e_2 \wedge \neg e_2 <_E e_1.$$

Then the thirteen relations discussed in Allen (1983) can be defined as:

DEFINITION 3.

$$e_1\ b\ e_2 =_{\text{def}} e_1 < e_2 \wedge \neg e_1 : e_2;$$

$$e_1\ :\ e_2 =_{\text{def}} e_1 : e_2;$$

$$e_1\ o\ e_2 =_{\text{def}} e_1 \bigcirc e_2 \wedge e_1 <_B e_2 \wedge e_1 <_E e_2;$$

$$e_1\ s\ e_2 =_{\text{def}} e_1 =_B e_2 \wedge e_1 <_E e_2;$$

$$e_1\ f\ e_2 =_{\text{def}} e_2 <_B e_1 \wedge e_1 =_E e_2;$$

$$e_1\ d\ e_2 =_{\text{def}} e_2 <_B e_1 \wedge e_1 <_E e_2;$$

$$e_1\ \equiv\ e_2 =_{\text{def}} e_1 =_B e_2 \wedge e_1 =_E e_2;$$

$$e_1\ bi\ e_2 =_{\text{def}} e_2\ b\ e_1;$$

$$e_1\ {:}i\ e_2 =_{\text{def}} e_2\ :\ e_1;$$

$$e_1\ oi\ e_2 =_{\text{def}} e_2\ o\ e_1;$$

$$e_1 \; si \; e_2 =_{\text{def}} e_2 \; s \; e_1;$$

$$e_1 \; fi \; e_2 =_{\text{def}} e_2 \; f \; e_1;$$

$$e_1 \; di \; e_2 =_{\text{def}} e_2 \; d \; e_1;$$

$$e_1 \bowtie e_2 =_{\text{def}} e_1 \; (b \; bi : :i \; d \; di \; f \; fi \; o \; oi \; s \; si \equiv) \; e_2.$$

At this stage we have actually arrived at an axiomatisation of Allen's event logic (Allen 1983), as shown in the following theorem:

THEOREM 2. *Allen's transitivity table (table 1) can be derived from* R1–R10, *and Definitions 2 and 3.*

Proof. The proof consists of proofs for all entries in table 1. For example, for entry (1, 1) we need to prove $e_1 \; b \; e_2 \wedge e_2 \; b \; e_3 \rightarrow e_1 \; b \; e_3$. *Proof.* Suppose $e_1 \; b \; e_2$ and $e_2 \; b \; e_3$. Then by Definition 3 we have $e_1 < e_2$ and $e_2 < e_3$. So by R2 we have $e_1 < e_3$. Now, by R2, $e_2 \bigcirc e_2$. So we have $e_1 < e_2$, $e_2 \bigcirc e_2$, and $e_2 < e_3$. So by R8, we have $\neg e_1 : e_3$. Since $e_1 < e_3$ and $\neg e_1 : e_3$, so $e_1 \; b \; e_3$ by Definition 3. □ The rest of the entries in table 1 can be proved analogously. □

We have now obtained another axiomatisation of Allen's logic of events, which we shall call "the Russell axiomatisation". What is the formal relationship between the Allen-Hayes axiomatisation and the Russell axiomatisation? We can show that the former is stronger than the latter. The reason is this. On the one hand, some theorems (including axioms) in the Allen-Hayes axiomatisation are not provable in the Russell axiomatisation. For example, A1 says that every event is abutted by an event at both ends, and A5 asserts that for every pair of meeting events there is an event which begins at the same time as the first of the pair and ends at the same time as the second of the pair. These are not derivable from the Russell axiomatisation. On the other hand, all the axioms in the Russell axiomatisation can be derived from the Allen-Hayes axiomatisation. To see this, let us first look at a result from Tsang (1987). Tsang defines the relations "subevent (*in*)", "intersection (!)", and "union (∪)" between events as:

$$e_1 \; in \; e_2 =_{\text{def}} \forall e_3 (e_3 \bigcirc e_1 \rightarrow e_3 \bigcirc e_2);$$

$$e = e_1 ! e_2 =_{\text{def}} e_1 \bigcirc e_2 \wedge e \; in \; e_1 \wedge e \; in \; e_2' \wedge$$
$$\forall e' (e' \; in \; e_1 \wedge e' \; in \; e_2 \rightarrow e' \; in \; e);$$

$$e = e_1 \cup e_2 =_{\text{def}} e_1 \; in \; e \wedge e_2 \; in \; e \wedge \forall e' \; (e' < e_1 \wedge e' < e_2 \rightarrow e' < e) \wedge$$
$$\forall e' (e_1 < e' \wedge e_2 < e' \rightarrow e < e').$$

He then proves the following lemma:

Lemma 1 (Tsang 1987). *Define:*

$$e_1 < e_2 =_{\text{def}} e_1 : e_2 \vee \exists e_3(e_1 : e_3 : e_2);$$

$$e_1 \bigcirc e_2 =_{\text{def}} \neg e_1 < e_2 \wedge \neg e_2 < e_1.$$

Then: if "$:$" *satisfies* A1–A5 *then* "$<$" *and* "$\bigcirc$" *satisfy* R1–R7 + E8–E11, *where*:

E8: $\forall e \exists e_1(e_1 < e \wedge \neg \exists e_2(e_1 < e_2 < e))$;
E9: $\forall e \exists e_1(e < e_1 \wedge \neg \exists e_2(e < e_2 < e_1))$;
E10: $\forall e_1 e_2 \in E(e_1 \bigcirc e_2 \rightarrow e_1 ! e_2 \in E)$;
E11: $\forall e_1 e_2 \in E(e_1 \cup e_2 \in E)$.[6]

With Lemma 1, we can prove the following theorem:

Theorem 3. *Define:*

$$e_1 < e_2 =_{\text{def}} e_1 : e_2 \vee \exists e_3(e_1 : e_3 : e_2);$$

$$e_1 \bigcirc e_2 =_{\text{def}} \neg e_1 < e_2 \wedge \neg e_2 < e_1;$$

$$e_1 <_B e_2 =_{\text{def}} \exists e_3 e_4(e_3 : e_1 \wedge e_3 : e_4 : e_2);$$

$$e_1 <_E e_2 =_{\text{def}} \exists e_3 e_4(e_2 : e_4 \wedge e_1 : e_3 : e_4).$$

Then: *if* "$:$" *satisfies* A1–A5 *then* "$<$", "$\bigcirc$", "$:$", "$<_B$" *and* "$<_E$" *satisfy* R1–R10.

Proof. Suppose that "$:$" satisfies A1–A5. Then R1–R7 follows directly from Lemma 1. The proof for R8–R10 needs to make use of E10. □

It is thus clear that the Allen-Hayes axiomatisation is stronger than the Russell axiomatisation. We have seen in Section 2.1 above that the Allen-Hayes axiomatisation fails the finite-closed-world assumption. What about the Russell axiomatisation?

It is evident that the axioms R1–R10 do not require that there are an infinite number of events. So the Finiteness Criterion is met. But do these axioms and the definitions of the other relations require events outside the closed world? It is clear that Definitions 2 and 3 do not introduce any extra events. The only possibility lies in R8–R10. But if R8–R10 are indeed true in the world, then they do not require any additional events, and thus the axiomatisation will satisfy the Closedness Criterion. We shall say that the world is *saturated* if R8–R10 hold in it, and *unsaturated* otherwise. Thus, if the world is saturated, then the Russell axiomatisation will satisfy the

[6] E8 and E9 state that every event meets and is met by another event; E10 says that if two events overlap then their intersection is also an event; and E11 asserts that the union of two events is always an event.

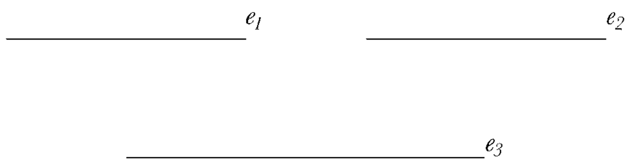

Figure 2. Events in an unsaturated world.

Closedness Criterion. But if the world is not saturated, then the Russell axiomatisation will violate the Closedness Criterion, because it will have to assume that there are events apart from those in the world. For example, consider the world depicted in figure 2 (same as figure 1) below. In that world, e_1 begins before e_3, but there are no events which overlap e_1 and which precede e_3. But R9 would require that such an event exist in that world, and would therefore violate the Closedness Criterion.

Suppose that a world is finite, but is unsaturated (such as the one in figure 2). What axiomatisation can we have for Allen's logic which does not violate the Criteria of Finiteness and Closedness? It is this question that I shall turn to in the next section.

2.3. The Axiomatisation Based on the Thomason Event Structure

When a world is unsaturated, it is not the case that R8–R10 are true. For example, in figure 2 the world has only three events in it. It is true that e_1 begins before e_3, and but there is no event e such that $e \bigcirc e_1$ and $e < e_3$ (the existence of such an event is required by R9). It is also true that e_1 does not meet e_2, though this situation satisfies R8. So R8–R10 do not hold in this (unsaturated) world. But if we discard R8–R10, then there will be no axioms which relate "$<$" and "$\bigcirc$" on the one hand and "$:$", "$<_B$" and "$<_E$" on the other; and we will not be able to reason about events which involve all these five relations. So when the world is unsaturated we need a different set of axioms. In an unsaturated world, we cannot define the relations "$:$", "$<_B$" and "$<_E$" in terms of "$<$" and "$\bigcirc$"; so they have to be treated as primitive relations. And these relations must satisfy certain intuitive conditions. Thomason (1989) has provided us with just the set of axioms we require, as shown in the following definition:[7]

DEFINITION 4. $\Sigma_T = \langle E, <, \bigcirc, :, <_B, <_E \rangle$ *is the Thomason event structure. It consists of a non-empty set E of events, together with the five temporal relations*:

[7] The definition is taken from Thomason (1989), with one modification: Thomason (1989) does not treat "$\bigcirc$" as a primitive relation. Instead, he would define it as: $e_1 \bigcirc e_2 =_{def} \neg e_1 < e_2 \wedge \neg e_2 < e_1$. Here I have regarded "$\bigcirc$" as a primitive relation and the above definition as an axiom. This slight modification permits great concision in the exposition of the ideas in this section as well as in Section 3.2.

"*before* ($<$)", "*overlap* ($\bigcirc$)", "*meets* ($:$)", "*begins-before* ($<_B$)", *and* "*ends-before* ($<_E$)". *These relations are assumed to satisfy the following axioms* T1–T13*:*

T1: $\neg e_1 < e_1$;
T2: $e_1 < e_2 \wedge e_3 < e_4 \rightarrow e_1 < e_4 \vee e_3 < e_2$;
T3: $e_1 <_B e_2 \rightarrow \neg e_2 <_B e_1$;
T4: $e_1 <_E e_2 \rightarrow \neg e_2 <_E e_1$;
T5: $e_1 <_B e_2 \rightarrow e_3 <_B e_2 \vee e_1 <_B e_3$;
T6: $e_1 <_E e_2 \rightarrow e_3 <_E e_2 \vee e_1 <_E e_3$;
T7: $e_1 < e_2 \wedge \neg e_1 < e_3 \rightarrow e_3 <_B e_2$;
T8: $e_1 < e_2 \wedge \neg e_3 < e_2 \rightarrow e_1 <_E e_3$;
T9: $e_1 : e_2 \rightarrow e_1 < e_2$;
T10: $e_1 : e_2 \wedge e_1 < e_3 \rightarrow \neg e_3 <_B e_2$;
T11: $e_1 : e_2 \wedge e_3 < e_2 \rightarrow \neg e_1 <_E e_3$;
T12: $\neg e_1 <_E e_3 \wedge \neg e_3 <_E e_1 \wedge \neg e_2 <_B e_4 \wedge \neg e_4 <_B e_2 \rightarrow (e_1 : e_2 \leftrightarrow e_3 : e_4)$;
T13: $e_1 \bigcirc e_2 \leftrightarrow \neg e_1 < e_2 \wedge \neg e_2 < e_1$.

The intuitions behind T1, T3, T4, T9, and T13 are obvious: we all know that events do not precede themselves (T1); that if one event begins before (or ends before) another event then it is not the case that the latter begins before (or ends before) the former (T3 and T4); that if one event meets another then it is also before it (T9); and that if two events overlap then neither precedes the other (T13). The remaining axioms have to be, and can be, justified in terms of the intended meanings of "$<$", "$:$", "$<_B$", and "$<_E$". Thomason (1989) does not answer the question how these less obvious axioms are motivated. But it is reasonable to think that these axioms are needed in order to ensure that instants constructed from the events have the desired properties, such as that they form a linear ordering (cf. Lemma 2 and Theorem 6 in Section 3.1 below).

It is perhaps helpful to discuss briefly Thomason's motivations for providing the Thomason event structure. Like Russell, Thomason is interested in how instants of time can be constructed on the basis of events and their temporal relationships. Russell's construction of instants as maximal sets of overlapping events, which we shall see in Section 3.2 below, only ensures that the instants constructed form a linear ordering. But it is difficult to formulate conditions on the event structure which will ensure that the resultant instants form a continuum, isomorphic to real numbers. So Thomason provides another way of constructing instants: each instant divides the events into a past, present, and a future (cf. Section 3.1 below), a proposal inspired by Walker (1947). To define instants this way, Thomason (1984) finds that it is desirable to treat "$<$", "$:$", "$<_B$", and "$<_E$" all as primitive relations, and he succeeds in discovering the conditions on the event structure which ensure that the instants form a continuum. Thomason (1984) assumes that there are an infinite number of events, and that the set of axioms is consequently different from T1–T13 given above. But in Thomason (1989) he investigates the cases where there are only a finite number of events. There he argues that a sentient being is finite in nature and therefore can only observe a finite number of events on any occasion. Thomason wants to see how instants can be constructed in such cases. But he has to formulate the axioms for events in such cases first, and he provides T1–T13. In Section 3.1 below I discuss Thomason's way of constructing instants out of a finite number of events.

Given the five temporal relations contained in the Thomason event structure, we can define other temporal relations in terms of these five, as in Definitions 2 and 3. We can also show that Allen's logic of events is axiomatisable using the Thomason event structure:

THEOREM 4. *Allen's transitivity table (table 1) can be derived from* T1–T13, *and Definitions 2 and 3.*

Proof. The proof consists of proofs for all entries in table 1. For example, for entry (1, 1) we need to prove $e_1\ b\ e_2 \wedge e_2\ b\ e_3 \rightarrow e_1\ b\ e_3$. *Proof.* Suppose $e_1\ b\ e_2$ and $e_2\ b\ e_3$. Then by Definition 3 we have $e_1 < e_2$ and $e_2 < e_3$. By T1 we have $\neg e_2 < e_2$. So we have $e_2 < e_3$ and $\neg e_2 < e_2$; so by T7 $e_2 <_B e_3$. Thus we now have $e_1 < e_2$ and $e_2 <_B e_3$. It is to be shown that $e_1 < e_3$ and $\neg e_1 : e_3$. Suppose the contrary, then either $\neg e_1 < e_3$ or $e_1 : e_3$. If $\neg e_1 < e_3$, then $e_3 < e_B\, e_2$ by T7 (because $e_1 < e_2$ and $\neg e_1 < e_3$); so we have $e_2 <_B e_3$ and $e_3 <_B e_2$: contradicting T3. If $e_1 : e_3$, then we have $e_1 : e_3$ and $e_1 < e_2$; so by T10 $\neg e_2 <_B e_3$, but this contradicts the fact that $e_2 <_B e_3$ according to T3. Thus it must be true that $e_1 < e_3$ and $\neg e_1 : e_3$. So by Definition 3 $e_1\ b\ e_3$ must be true. This completes the proof for entry (1, 1) in table 1.□ The rest of the entries in table 1 can be proved analogously. □

Having obtained the Thomason axiomatisation of Allen's logic of events, let us now discuss the formal relationship between it and the Russell axiomatisation. We can show that, if the world is saturated, the Thomason axiomatisation and the Russell axiomatisation will be (strongly) equivalent. This is expressed in the following theorem:

THEOREM 5. *If* R8–R10 *hold*, *then* R1–R7 *and* T1–T13 *are mutually derivable.*

Proof. The proof consists of showing, case by case, that R1–R7 are derivable from T1–T13 and vice versa when R8–R10 are true. For example, we can show that R2 follows from T1–T13. *Proof.* Suppose $e_1 < e_2$ and $e_2 < e_3$. Then by T2 $e_1 < e_3$ or $e_2 < e_2$. The latter contradicts T1 and so must be discarded. So $e_1 < e_3$ must be true. □ As another example, let us show that T3 follows from R1–R7. *Proof.* Suppose $e_1 <_B e_2$. Then by R9 there is e_3 such that $e_1 \bigcirc e_3$ and $e_3 < e_2$. Suppose now $e_2 <_B e_1$. Then by R9 there is e_4 such that $e_2 \bigcirc e_4$ and $e_4 < e_1$. So we have $e_3 < e_2$, $e_2 \bigcirc e_4$, and $e_4 < e_1$; so by R6, $e_3 < e_1$. So we have $e_1 \bigcirc e_3$ and $e_3 < e_1$. But this contradicts R5, so it must be the case that $\neg e_2 <_B e_1$. □ □

According to the above theorem, T1–T13 can be derived from R1–R10. But we cannot derive R8–R10 from T1–T13. So the Russell axiomatisation is stronger than the Thomason axiomatisation.

Does the Thomason axiomatisation satisfy the finite-closed-world assumption? The answer is positive. This is because i) the axioms T1–T13 do not require the presence of an infinite number of events; and ii) the definitions of the thirteen relations used in Allen's event logic do not require events outside E of $\Sigma_T = \langle E, <, \bigcirc, :, <_B, <_E \rangle$. Notice that i) and ii) are true no matter whether the world is saturated or not.

3. Time in the Finite and Closed World

In the previous section I developed two axiomatisations of Allen's logic of events. Both the Russell axiomatisation and the Thomason axiomatisation satisfy the Finiteness Criterion. But the former satisfies the Closedness Criterion only when the world is saturated, whereas the latter does so even when the world is unsaturated. In this section I focus on how instants can be constructed from events in a finite and closed world. I deal with instants in an unsaturated world first.

3.1. Instants in the unsaturated world

When the world is not saturated, it is best modelled by the Thomason event structure $\Sigma_T = \langle E, <, \bigcirc, :, <_B, <_E \rangle$. We want to construct "instants" from $\Sigma_T = \langle E, <, \bigcirc, :, <_B, <_E \rangle$ such that each "instant" represents a "present", or a state of affairs of the world (and hence divides the world into "past", "present", and "future").

Let us consider figure 3, which describes an unsaturated world. Intuitively, there are five different "instants", or states of affairs. How can we construct such instants from events in this unsaturated world?

Thomason (1989) provides one such construction. His definition of instants is similar to that of Walker (1947). Each Thomason instant is also called a "cut", a term coined by Dedekind. Informally, each Thomason instant divides the events into three classes: events in the past, those in the present, and those in the future. The formal definition is given below (this definition is taken from Thomason 1984):

Definition 5. *A cut in* $\Sigma_T = \langle E, <, \bigcirc, :, <_B, <_E \rangle$ *is a triple* (P, C, F) *satisfying*:

CUT.1: $C = E - (P \cup F)$;

CUT.2: $e_1 \in P \wedge e_2 \in F \rightarrow e_1 < e_2 \wedge \neg e_1 : e_2$;

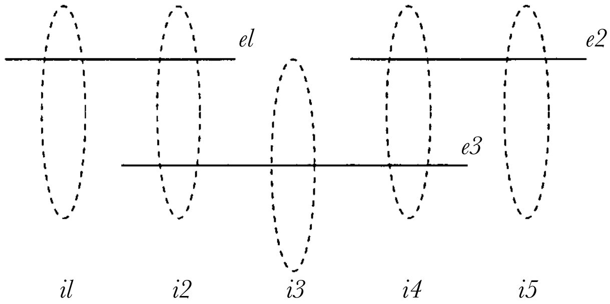

Figure 3. Instants in an unsaturated world.

CUT.3: $e_1 < e_2 \rightarrow e_1 \in P \vee e_2 \in F$;
CUT.4: $e_1 \in P \wedge e_2 \notin P \rightarrow e_1 <_E e_2$;
CUT.5: $e_1 \in F \wedge e_2 \notin F \rightarrow e_2 <_B e_1$;
CUT.6: $C = \varnothing \rightarrow P \neq \varnothing \wedge F \neq \varnothing$.

CUT.1–CUT.6 are self-explanatory, so we will not give their English translations here. It is easy to see that i_1 to i_5 in figure 2 are all cuts according to the definition.[8]

One cut is said to be *before* another if either the past of the one is included in the past of the other, or the future of the one includes the future of the other:

Definition 6. $(P_1, C_1, F_1) < (P_2, C_2, F_2)$ *if and only if* $P_1 \subset P_2$ *or* $F_2 \subset F_1$.

From Definitions 5 and 4 it is easy to obtain:

Lemma 2. Thomason 1989. *If* (P_1, C_1, F_1) *and* (P_2, C_2, F_2) *are two cuts in* Σ_T *then*:

i) $P_1 \subseteq P_2 \vee P_2 \subseteq P_1$;
ii) $P_1 \subset P_2 \rightarrow F_2 \subseteq F_1$;
iii) $F_1 \subseteq F_2 \vee F_2 \subseteq F_1$;
iv) $F_1 \subset F_2 \rightarrow P_2 \subseteq P_1$.

Let $I(\Sigma_T)$ be the set of cuts of Σ_T and "$<$" the precedence relation on $I(\Sigma_T)$. Then the structure $\tau(\Sigma_T) = \langle I(\Sigma_T), < \rangle$ is called the *derived cut structure* from Σ_T. Using Lemma 2 we can show that the cuts form a linear series:

Theorem 6. *If* $\Sigma_T = \langle E, <, \bigcirc, :, <_B, <_E \rangle$ *satisfies* T1–T13, *then* $\tau(\Sigma_T) = \langle I(\Sigma_T), < \rangle$ *is a strict linear ordering. I.e.* $\tau(\Sigma_T)$ *satisfies the following conditions*:

i) $c_1 < c_2 \rightarrow \neg c_2 < c_1$;
ii) $c_1 < c_2 \wedge c_2 < c_3 \rightarrow c_1 < c_3$;
iii) $c_1 \neq c_2 \rightarrow c_1 < c_2 \wedge c_2 < c_1$.

[8] Each ellipse presents a cut (P, C, F), where P consists of all the events lying to its left, F of all those lying to its right, and C of all those which are enclosed in it.

Proof. We prove i)–iii) one by one:

i) Suppose both $c_1 < c_2$ and $c_2 < c_1$. Then by Definition 6 we have $P_1 \subset P_2 \vee F_2 \subset F_1$, and $P_2 \subset P_1 \vee F_1 \subset F_2$. There are four possible combinations.

i.a) $P_1 \subset P_2$ and $P_2 \subset P_1$: contradiction.

i.b) $F_2 \subset F_1$ and $F_1 \subset F_2$: contradiction.

i.c) $P_1 \subset P_2$ and $F_1 \subset F_2$. Because $P_1 \subset P_2$, so by Lemma 2.ii) $F_2 \subseteq F_1$. So we have $F_1 \subset F_2$ and $F_2 \subseteq F_1$: contradiction.

i.d) $F_2 \subset F_1$ and $P_2 \subset P_1$. Because $F_2 \subset F_1$, so by Lemma 2.iv) $P_1 \subseteq P_2$. So we have $P_2 \subset P_1$ and $P_1 \subseteq P_2$: contradiction.

Since these four cases are all ruled out, the assumption that both $c_1 < c_2$ and $c_2 < c_1$ hold must be false. Therefore, $c_1 < c_2 \rightarrow \neg c_2 < c_1$ must be true.

ii) Suppose $c_1 < c_2$ and $c_2 < c_3$. Then by Definition 6 $P_1 \subset P_2 \vee F_2 \subset F_1$, and $P_2 \subset P_3 \vee F_3 \subset F_2$. There are four possible combinations.

ii.a) $P_1 \subset P_2$ and $P_2 \subset P_3$. Then $P_1 \subset P_3$, so $c_1 < c_3$.

ii.b) $F_2 \subset F_1$ and $F_3 \subset F_2$. Then $F_3 \subset F_1$, so $c_1 < c_3$.

ii.c) $P_1 \subset P_2$ and $F_3 \subset F_2$. Because $P_1 \subset P_2$, so by Lemma 2.ii) $F_2 \subseteq F_1$. So $F_3 \subset F_1$, and hence $c_1 < c_3$.

i.d) $F_2 \subset F_1$ and $P_2 \subset P_3$. Because $F_2 \subset F_1$, so by Lemma 2.iv) $P_1 \subseteq P_2$. So $P_1 \subset P_3$, thus $c_1 < c_3$.

Since these four cases all lead to $c_1 < c_3$, so $c_1 < c_2 \wedge c_2 < c_3 \rightarrow c_1 < c_3$ must be true.

iii) Suppose $c_1 \neq c_2$. Then $(P_1, C_1, F_1) \neq (P_2, C_2, F_2)$. So either $P_1 \neq P_2$ or $F_1 \neq F_2$ or $C_1 \neq C_2$.

iii.a) $P_1 \neq P_2$. Then $P_1 \subset P_2$ or $P_2 \subset P_1$ (cf. Lemma 2.i). So $c_1 < c_2$ or $c_2 < c_1$.

iii.b) $F_1 \neq F_2$. Then $F_1 \subset F_2$ or $F_2 \subset F_1$ (cf. Lemma 2.iii). So $c_1 < c_2$ or $c_2 < c_1$.

iii.c) $C_1 \neq C_2$. Because $C_1 = E - (P_1 \cup F_1)$, $C_2 = E - (P_2 \cup F_2)$, so $P_1 \cup F_1 \neq P_2 \cup F_2$. Thus $\neg(P_1 = P_2 \wedge F_1 = F_2)$, which is $P_1 \neq P_2$ or $F_1 \neq F_2$. In both cases we have $c_1 < c_2$ or $c_2 < c_1$ (cf. iii.a) and iii.b) above). So $c_1 < c_2$ or $c_2 < c_1$ follow from $C_1 \neq C_2$.

iii.a)–iii.c) above together show that $c_1 < c_2 \vee c_2 < c_1$ follows from $c_1 \neq c_2$.

□

If an event belongs to the set C of a cut (P, C, F), then we shall say that the event *goes on* at that cut. Having shown that the cuts form a strict linear ordering, we can also show that every event goes on at least one cut, and that two overlapping events always have at least one cut at which they both go on. Since an interval is usually defined as a convex set of instants or cuts (see van Benthem 1983), we can also show that every event goes on at an interval of time. Thus, the instants (cuts) constructed via the Thomason construction do satisfy the typical requirements we expect of time instants (Russell 1926, 1956, Lin 1991). But since these results are not essential to the present paper I shall omit the details.

In Section 2.3 above, I briefly discussed the motivation of the Thomason event structure (see Definition 4 above). I mentioned that Thomason (1989) wants to see how instants are constructed out of a finite number of events. But such instants, such as those in figure 3, are not really instantaneous, i.e. not genuinely durationless. Thomason (1989) also considers the question how the concept of real instants is obtained from instants such as the ones in figure 3. The idea is this. In figure 3, there are five time instants. They are instantaneous as far as this particular world is concerned. They represent various states of affairs of this particular world. Of course these five instants are not really "instantaneous": each of them is in fact an interval of time. This can be explained as follows. Apart from the events depicted in figure 3, there are of course other events going on. If we take those events into consideration, then an instant in figure 3 will become a series of instants (i.e. an interval) of the bigger world. If we take all the events in the universe into consideration, then the instants constructed out of all these events will be truly "instantaneous". For the details, see Thomason 1989.

3.2. Instants in the saturated world

The preceding section presents a way of constructing instants in an unsaturated world where R8–R10 may not hold. This way of constructing instants surely also works in a saturated world where R8–R10 are true. We have seen how the axiomatisation of Section 2.2 can be simplified by using the Russell event structure when the world is saturated (Section 2.3, Theorem 5). So we wonder whether a simpler construction of instants (than the Thomason construction) can also be found when the world is saturated.

Russell (1926, 1956) offers a way of constructing instants. His motivation is that he wants to show that our conception of abstract instants is derived from the events and the temporal relationships we perceive. Russell defines instants as maximal sets of pairwise overlapping events. He shows that instants thus constructed have the properties we normally require of instants of time, such as that they form a strict linear ordering. So Russell's theory of time is part of his "Logical Atomicism".

Formally, Russell's definition of instants can be put as follows (the definition is taken from Kamp 1979 and slightly modified):

DEFINITION 7. *Let* $\Sigma_R = \langle E, <, \bigcirc, :, <_B, <_E \rangle$ *be an event structure. An instant i is a subset of E such that*:

i) $\forall e_1, e_2 \in i\ (e_1 \bigcirc e_2)$;
ii) $\forall e_1 \notin i\ (\exists e_2 \in i \neg e_1 \bigcirc e_2)$.

We denote the set of instants of Σ_R as $I(\Sigma_R)$. We say that an instant is "before" another when some member of the former precedes some member of the latter:

Definition 8. Kamp 1979. *Let i_1 and i_2 be any two instants of $I(\Sigma_R)$. Then*: $i_1 < i_2 =_{\text{def}} \exists e_1 \in i_1\, \exists e_2 \in i_2\ (e_1 < e_2)$.

We then call $\tau(\Sigma_R) = \langle I(\Sigma_R), < \rangle$ the *instant structure* derived from Σ_R.

The properties of the instant structure $\tau(\Sigma_R) = \langle I(\Sigma_R), < \rangle$, including linearity, density, discreteness, and the existence of instants etc. have been previously discussed (Kamp 1979, 1980, Lin 1991). Here I want to show that the Russell construction and the Thomason construction are closely related: if the world is finite and saturated, the Russell construction is equivalent to the Thomason construction, that is, the instant structures produced by the two constructions are isomorphic.

First, we show that, in a saturated world, each Russell instant defines a Thomason instant:

Theorem 7. *Assume that $\Sigma_T = \langle E, <, \bigcirc, :, <_B, <_E \rangle$ is saturated, that is, assume that Σ_T satisfies* T1–T13 *and* R8–R10. *Let C be any Russell instant. Let P be the set of all events which precede C, i.e.* $P = \{e_1 \mid \exists e_2 \in C\ e_1 < e_2\}$, *F the set of all events which succeed C, i.e.* $F = \{e_1 \mid \exists e_2 \in C\ e_2 < e_1\}$. *Then* (P, C, F) *is a Thomason cut.*

Proof. It suffices to show that (P, C, F) satisfies CUT.1–CUT.6. (Bear in mind that R1–R7 are now also satisfied, cf. Theorem 5.)

i) Let $e \in E$ and $e \notin C$, then $\exists e' \in C(\neg e' \bigcirc e)$. So $e' < e \vee e < e'$ by R7. Because $e \notin C$, so $e \in P$ or $e \in F$. Thus any event of E is in either P, or F, or C. It is obvious that these three cases are mutually exclusive, so $C = E - (P \cup F)$.

ii) Let $e_1 \in P$ and $e_2 \in F$. Then there are e_3 and e_4 in C such that $e_1 < e_3$ and $e_4 < e_2$. Because, $e_3, e_4 \in C$, so $e_3 \bigcirc e_4$. So $e_1 < e_2$ by R6, and $\neg e_1 : e_2$ by R8.

iii) Let $e_1 < e_2$. It is true that either $e_1 \in P$ or $e_1 \in C$ or $e_1 \in F$. If $e_1 \in P$, then it is true that $e_1 \in P \vee e_2 \in F$. If $e_1 \in C$, then $e_2 \in F$ because $e_1 < e_2$; so it is also true that $e_1 \in P \vee e_2 \in F$. If $e_1 \in F$, then there is $e_3 \in C$ such that $e_3 < e_1$; because $e_1 < e_2$, so $e_3 < e_2$; so $e_2 \in F$; therefore it is still true that $e_1 \in P \vee e_2 \in F$. Thus $e_1 \in P \vee e_2 \in F$ follows from $e_1 < e_2$.

iv) Let $e_1 \in P$ and $e_2 \notin P$. Then either $e_2 \in C$ or $e_2 \in F$. Consider the former alternative. There must be an event e in C such that $e_1 < e$. Because e and e_2 are both in C, so $e \bigcirc e_2$. Thus $e_1 < e$ and $e \bigcirc e_2$; so $e_1 <_E e_2$. Now consider the latter alternative. Because $e_1 \in P$ and $e_2 \in F$, so $e_1 < e_2$ by ii) above. Therefore $e_1 <_E e_2$. Thus $e_1 <_E e_2$ is true.

v) This can be proved in a similar way to iv).

vi) Every Russell instant is not empty. To see this, let C be a Russell instant and suppose $C = \varnothing$. Because E is non-empty, so there must be an event $e \in E$ such that $e \notin C$. By Definition 7.ii) there must be an event $e' \in C$. But this contradicts $C = \varnothing$. So C cannot be empty. As a result, $C = \varnothing \rightarrow P \neq \varnothing \wedge F \neq \varnothing$ is trivially true. □

Next, we show that, when the world is saturated and finite, each Thomason instant also defines a Russell instant:

THEOREM 8. *Assume that* $\Sigma_T = \langle E, <, \bigcirc, :, <_B, <_E \rangle$ *is finite and saturated, that is, assume that* Σ_T *satisfies* T1–T13 *and* R8–R10, *and that E is finite. Let* (P, C, F) *be any Thomason instant. Then C is a Russell instant.*

Proof. It suffices to show that C is a maximal pairwise overlapping set of events. (Bear in mind that R1–R7 are now also satisfied, cf. Theorem 5.)

i) We first show that members of C are pairwise overlapping. Let $e_1, e_2 \in C$, and suppose that not $e_1 \bigcirc e_2$. Then either $e_1 < e_2$ or $e_2 < e_1$ by R7. If $e_1 < e_2$ then by CUT.3 $e_1 \in P$ or $e_2 \in F$: contradicting the fact that e_1 and e_2 are both in C. The case with $e_2 < e_1$ can be treated analogously. Thus it must be true that $e_1 \bigcirc e_2$.

ii) We now show that C is maximal. Let $e \notin C$. We must show that there is $x \in C$ such that $\neg x \bigcirc e$. So we need to show in the first place that C is non-empty (this is because a Russell instant is never empty, see vi) in the proof of Theorem 7 above). Suppose the contrary, i.e. $C = \varnothing$. Then by CUT.6 $P \neq \varnothing \wedge F \neq \varnothing$. Because E is finite, so P and F must be finite; so there must be an event e_1 which is the last to end in P. Similarly, there must be an event e_2 which is the first to begin in F. By CUT.2 $e_1 < e_2$ and $\neg e_1 : e_2$. Thus by R8 there are two events e_3 and e_4 such that $e_3 \bigcirc e_4$, $e_1 < e_4$, and $e_3 < e_2$. It is obvious that $e_3 <_B e_2$ and $e_1 <_E e_3$. Now e_3 must be in C; otherwise if $e_3 \in F$ then e_2 would not be the first to begin in F, and if $e_3 \in P$ then e_1 would not be the last to end in P. Similarly, because $e_1 <_E e_4$ and $e_4 <_B e_2$, e_4 must also be in C. But these contradict $C = \varnothing$. Therefore, C cannot be empty.

We continue to show that there must be an event $x \in C$ such that $\neg x \bigcirc e$. Because $e \notin C$, so either $e \in P$ or $e \in F$. Let us first consider the case where $e \in P$. Because e_1 is the last to end in P, so e_1 cannot end before e, i.e. $\neg e_1 <_E e$. Because $e_1 < e_4$, it must be that $e < e_4$. So $e_4 \in C$ and $\neg e_4 \bigcirc e$. Similarly, if $e \in F$, then $e_3 < e$; so $e_3 \in C$ and $\neg e_3 \bigcirc e$. So there is always an event in C which does not overlap e. This proves that C must be maximal. □

Since, in a finite and saturated world, a Russell instant defines a Thomason cut and vice-versa, the Russell instant structure and the Thomason cut structure are isomorphic:

THEOREM 9. *Let $\Sigma_T = \langle E, <, \bigcirc, :, <_B, <_E \rangle$ be finite and saturated. Then the Russell instant structure and the Thomason cut structure are isomorphic.*

Proof. Theorem 9 follows directly from Theorems 7 and 8. □

4. SUMMARY

In this paper we examined the Allen-Hayes axiomatisation of the logic of events, and developed two other axiomatisations based on Russell (1926, 1956) and Thomason (1984, 1989). The first is stronger than the second, which is in turn stronger than the third. The Allen-Hayes axiomatisation fails to meet the criteria of Finiteness and Closedness. The Russell and the Thomason axiomatisations both satisfy the Finiteness Criterion. But the Russell axiomatisation meets the Closedness Criterion only when the world is saturated. If the world is not saturated, then the Thomason axiomatisation is the only possibility of the three. But if the world is saturated, then the Thomason axiomatisation will be equivalent to the Russell axiomatisation.

We also studied two ways of constructing instants of time from the events in a finite and closed world. The Russell construction is limited to the case where the world is saturated, but the Thomason construction can also work when the world is unsaturated. But when the world is saturated (and finite), the Thomason construction is equivalent to the Russell construction.

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