RUSSELL'S PARADOX OF THE TOTALITY OF PROPOSITIONS

NINO B. COCCHIARELLA

Russell's "new contradiction" about "the totality of propositions" has been connected with a number of modal paradoxes. M. Oksanen has recently shown how these modal paradoxes are resolved in the set theory NFU. Russell's paradox of the totality of propositions was left unexplained, however. We reconstruct Russell's argument and explain how it is resolved in two intensional logics that are equiconsistent with NFU. We also show how different notions of possible worlds are represented in these intensional logics.

In Appendix B of his 1903 *Principles of Mathematics* (PoM), Russell described a "new contradiction" about "the totality of propositions" that his "doctrine of types" (as described in Appendix B) was unable to avoid.[1] In recent years this "new contradiction" has been connected with a number of modal paradoxes, some purporting to show that there cannot be a totality of true propositions,[2] or that even the idea of quantifying over the totality of propositions leads to contradiction.[3] A number of these claims have been discussed recently by Mika Oksanen and shown to be spurious relative to the set theory known as NFU.[4] In other words, if NFU is used instead of ZF as the semantical metalanguage for modal logic, the various "paradoxes" about the totality of propositions (usually construed as the totality of sets of possible worlds) can be seen to fail (generally because of the existence of a universal set and the failure of the general form of Cantor's power-set theorem in NFU). It is not clear, however, how Russell's own paradox about the totality of propositions is resolved on this analysis, and although Oksanen quoted Russell's description of the paradox in detail, he did not show how it is explained in NFU after his resolution of the other related modal paradoxes; in fact, it is not at all clear how this might be done in NFU.

[1] PoM, p. 527.
[2] See, e.g., Grim 1991, pp. 92f.
[3] See, e.g., Grim 1991, p. 119 and Jubien 1988, p. 307.
[4] See Oksanen 1999. NFU is a modified version of Quine's system NF. It was first described in Jenson 1968 and recently has been extensively developed in Holmes 1999.

Nordic Journal of Philosophical Logic, Vol. 5, No. 1, pp. 25–37.

One reason why Russell's argument is difficult to reconstruct in NFU is that it is based on the logic of propositions, and implicitly in that regard on a theory of predication rather than a theory of membership. A more appropriate medium for the resolution of these paradoxes, in other words, would be a formal theory of predication that is a counterpart to NFU. Fortunately, there are two such theories, λHST* and HST^*_λ, that are equiconsistent with NFU and that share with it many of the features that make it a useful framework within which to resolve a number of paradoxes, modal or otherwise.[5]

1. The Systems λHST* and HST^*_λ

The systems λHST* and HST^*_λ are designed to contain the essentials of a theory of logical form; namely, (1) the basic forms of predication (as represented in predicate logic), (2) propositional connectives, (3) quantifiers that reach into predicate as well as subject positions, and (4) nominalized predicates and propositional forms as abstract singular terms. These four components correspond to fundamental features of natural language, and each needs to be accounted for in any theory of logical form underlying natural language. A logic with these components amounts to a second-order predicate logic with nominalized predicates and propositional forms as abstract singular terms. The formal details of the grammar of such a logic are briefly as follows. We assume the availability of denumerably many individual variables and, for each positive integer n, denumerably many n-place predicate variables. (Propositional variables are construed as 0-place predicate variables.) We will use "x", "y", "z", with or without numerical subscripts to refer to individual variables, "F^n", "G^n", "H^n" to refer to n-place predicate variables, and "P", "Q", "P'", etc., as 0-place predicate (propositional) variables. (We usually drop the superscript when the context makes clear the degree of the predicate in question.) As primitive logical constants, we will use $\rightarrow$, $\neg$, $=$, $\forall$, and λ, and assume the others to be defined in the usual way. Complex predicates are generated from formulas by means of the λ-operator. Predicates are not singular terms, of course, but they can be transformed into such by nominalization, which formally we represent by the deletion of the parentheses and subject-positions that come with them in their role as predicates. For example, on the definition that follows, $F(x)$ and $G(x, y)$ are formulas in which F and

[5] See Cocchiarella 1986, chapters IV and VI for proofs of the connection of NFU with these systems. Also, see Cocchiarella 1985 for how these systems are related to Quine's systems NF and ML. For a discussion of the refutation of Cantor's power-set theorem in these systems, see Cocchiarella 1992.

G occur as predicates, but in $H(F)$ and $H(G)$, F and G occur as singular terms. In $F(F)$ and $G(F, G)$, F and G occur both as predicates and singular terms, though no one occurrence can be both as a predicate and as a singular term. For convenience, parentheses are required to occur as part of predicates only when they occur in a formula as a predicate.

In the following definition of a meaningful expression of type n, where n is a natural number, we use 0 to represent the type of *singular terms* (or just "terms" for short), 1 to represent the type of *formulas* (propositional forms), and $n+1$ to represent the type of *n-place predicate expressions*. For each natural number n, accordingly, we recursively define the *meaningful expressions of type n*, in symbols, ME_n, as follows:

1. every individual variable (or constant) is in ME_0, and every n-place predicate variable (or constant) is in both ME_{n+1} and ME_0;
2. if $a, b \in ME_0$, then $(a=b) \in ME_1$;
3. if $\pi \in ME_{n+1}$ and $a_1, \ldots, a_n \in ME_0$, then $\pi(a_1, \ldots, a_n) \in ME_1$;[6]
4. if $\varphi \in ME_1$ and $x_1, \ldots, x_n$ are pairwise distinct individual variables, then $[\lambda x_1 \ldots x_n \varphi] \in ME_{n+1}$;
5. if $\varphi \in ME_1$, then $\neg\varphi \in ME_1$;
6. if $\varphi, \chi \in ME_1$, then $(\varphi \rightarrow \chi) \in ME_1$;
7. if $\varphi \in ME_1$ and α is an individual or a predicate variable, then $(\forall a)\varphi \in ME_1$;
8. if $\varphi \in ME_1$, then $[\lambda\varphi] \in ME_0$; and
9. if $n > 1$, then $ME_n \subseteq ME_0$.

By clause (9), every predicate expression (without parentheses) is a singular term. This includes 0-place predicates but not formulas in general unless they are of the form $[\lambda\varphi]$, which can be taken as the nominalization of φ, and read as "that φ". For convenience, however, we shall write "$[\varphi]$" for "$[\lambda\varphi]$".[7]

Our initial goal, we have said, is to characterize a second-order predicate logic with nominalized predicate and propositional forms as abstract singular terms, because such a logic deals with the four important features of natural language described above. An important secondary goal, we believe, is that such a logic should contain (as a proper part) all of the theorems of second-order predicate logic, or rather at least all that are

---

[6] If $n=0$, we take $a_1, \ldots, a_n$ (and similarly $x_1, \ldots, x_n$) to be the empty sequence, resulting in this case in a 0-place predicate expression, which, as already noted, we take to be a formula.

[7] We could require that n be greater than 1 in clause (4)—in which case $[\lambda\varphi] \in ME_1$ would not follow—and then have clause (8) state that $[\varphi] \in ME_0$ when $\varphi \in ME_1$. But then general principles—such as (CP^*_λ) and $(\Box Ext^*)$ described below—that we want to apply to all n-place predicate expressions would have to be stated separately for $n=0$.

valid with respect to Henkin general models. (We cannot require that it contain all of the theorems that are valid in the "standard" models for second-order logic, of course, because such a logic is essentially incomplete by Gödel's (first) theorem.) This means in particular that we retain all of the theorems of classical propositional logic, and that all instances of the comprehension principle of "standard" second-order logic—i.e., instances in which abstract singular terms do not occur—should be provable.

It would be ideal if we could have all instances of the comprehension principle, even those in which nominalized predicates and propositional forms occur as abstract singular terms. But, assuming that the logic is not "free of existential presuppositions" for singular terms, such an unrestricted second-order logic—which is similar to the system of Frege's *Grundgesetze*[8]—is subject to Russell's paradox of predication, and therefore inconsistent. To preserve consistency, some restriction on the comprehension principle must be imposed. This of course is where the idea of stratification comes in. In a second-order predicate logic, where relations are involved, we need to base the restriction on homogeneous stratification, which in fact is also the restriction that applies to the "relations" (i.e., sets of ordered pairs) posited in NFU, and used by Oksanen in the resolution of some of the modal paradoxes involving belief as a relation.[9]

A formula or λ-abstract φ is *homogeneously stratified* (or just *h-stratified*) iff there is an assignment t of natural numbers to the terms and predicate expressions occurring in φ (including φ itself if it is a λ-abstract) such that (1) for all terms a, b, if $(a=b)$ occurs in φ, then $t(a)=t(b)$; (2) for all $n \geqslant 1$, all n-place predicate expressions π, and all terms $a_1, \ldots, a_n$, if $\pi(a_1, \ldots, a_n)$ is a formula occurring in φ, then (i) $t(a_i)=t(a_j)$, for $1 \leqslant i, j \leqslant n$, and (ii) $t(\pi)= t(a_1)+1$; (3) for $n \geqslant 1$, all individual variables $x_1, \ldots, x_n$, and formulas χ, if $[\lambda x_1 \ldots x_n \chi]$ occurs in φ, then (i) $t(x_i)=t(x_j)$, for $1 \leqslant i, j \leqslant n$, and (ii) $t([\lambda x_1 \ldots x_n \chi])=t(x_1)+1$; and (4) for all formulas χ, if $[\chi]$ (i.e., $[\lambda \chi]$) occurs in φ and $a_1, \ldots, a_k$ are all of the terms or predicates occurring in χ, then $t([\chi]) \geqslant \max\,[t(a_1), \ldots, t(a_k)]$.[10]

Now in the predicate logics λHST* and HST^*_λ the comprehension principle is stated as an identity, not as a biconditional. That is, in both systems the comprehension principle has the form

$$(\mathrm{CP}^*_\lambda) \quad (\exists F^n)([\lambda x_1 \ldots x_n \varphi]=F),$$

[8] Frege's expressions for value-ranges (*Wertverläufe*) were his formal counterparts of predicate nominalizations, i.e., formal counterparts of expressions such as "the concept *F*".

[9] See, e.g., p. 85 of Oksanen 1999.

[10] Clause (4), especially as it applies to nominalized formulas (i.e., nominalized propositional forms), was overlooked in earlier formulations of h-stratification, where our concern was with Russell's paradox of predication, and not with his paradox of the totality of propositions. We could require that $t([\chi])=\max\,[t(a_1), \ldots, t(a_k)]+1$, but our present formulation suffices to block Russell's argument.

where F does not occur (free) in φ.[11] By Leibniz's law (and λ-conversion) this implies the more usual biconditional form

$$(\text{CP*}) \quad (\exists F^n)(\forall x_1) \ldots (\forall x_n)(F(x_1, \ldots, x_n) \leftrightarrow \varphi).$$

The difference between these logics is that λHST* is not "free of existential presuppositions" for singular terms—i.e., λHST* retains the laws of "standard" first-order logic—and therefore every abstract singular term is assumed to denote a value of the bound individual variables, i.e.,

$$(\forall F^n)(\exists x)(x = F)$$

is provable λHST* (for all natural numbers n). What this means is that not all λ-abstracts can be taken to be well formed in λHST* (otherwise Russell's paradox of predication would be generated). Indeed, the principal restriction of λHST* is that to be well formed and allowed to occur in the formulas of λHST*, every λ-abstract must be h-stratified. In other words, in λHST* the only λ-abstracts that are allowed in (CP^*_λ) and (CP*) are those that are h-stratified, which effectively blocks even the formulation of Russell's paradox of predication.

In HST^*_λ, on the other hand, all λ-abstracts are well formed and allowed to occur in (CP^*_λ). But this is because HST^*_λ is "free of existential presuppositions" regarding singular terms, and hence the laws of first-order logic are qualified in HST^*_λ so that not all singular terms need be values of the bound individual variables.[12] Thus, although the λ-abstract for Russell's paradox,

$$(\exists F)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = F)$$

is both well formed and provable in HST^*_λ (but not in λHST* because it is not h-stratified)—i.e., although the Russell predicate stands for a concept in HST^*_λ—all that follows by Russell's argument is that as an abstract singular term it does not denote (a value of the bound individual variables). That is,

$$\neg(\exists y)([\lambda x(\exists G)(x = G \wedge \neg G(x))] = y)$$

is provable in HST^*_λ.[13]

---

[11] The "*" in "(CP^*_λ) and related principles is used as a reminder that nominalized predicates may occur in their instances, and hence that the principles go beyond what is posited in standard second-order predicate logic.

[12] There are vacuous singular terms in natural language of course, and hence such a qualification is appropriate and within our natural-language guidelines.

[13] The axiom schema regarding some, but not necessarily all, of the λ-abstracts that are consistently assumed to denote as abstract singular terms is ($\exists$/HSCP^*_λ), a description of which, along with that of the other axioms of these systems, can be found in Cocchiarella 1986 and in the Appendices of Cocchiarella 1992.

What this means is that what a λ-abstract denotes in HST^*_λ as a singular term (if it denotes anything at all) is not the concept that it stands for in its role as a predicate, because whereas the concept is always posited to "exist" (as a concept, i.e., as a value of the bound predicate variables), the λ-abstract must in some cases fail to denote (a value of the bound individual variables). Thus, whereas λHST* can be viewed as a reconstruction of the logic implicit in Russell's 1903 *Principles*—because, according to Russell, nominalized predicates denote the same properties and relations that predicates otherwise stand for—the system HST^*_λ cannot be so taken as well. It can be viewed as a reconstruction of Frege's *Grundgesetze*, however, because, according to Frege, the concepts and relations that are values of the predicate variables are "unsaturated" entities, whereas the entities denoted by abstract singular terms—e.g., Frege's expressions for value-ranges—are saturated objects. In addition, the idea that some concepts might have no object (extension) corresponding to them was in fact actually considered by Frege as one way to avoid the derivation of Russell's paradox in his system.[14]

There is also an alternative to Frege's "realism" regarding concepts (and propositions); namely, a form of conceptual realism where concepts are unsaturated cognitive capacities underlying our rule-following abilities in the use of the predicate phrases of natural language, rather than functions from objects to truth values. Nominalization in this framework amounts to a reification, or "*object* ification," of the truth conditions determined by the concepts that predicate expressions stand for—though in some cases, as in Russell's paradox, such truth conditions cannot be reified as objects. Where reification is possible, the result can be taken as an intensional object that is subject to identity conditions that vary depending on the kind of nonextensional contexts being considered in a given application of the logic. The conceptual counterpart of nominalizing a sentence can similarly be taken as a reification, or *object* ification, of the truth conditions of an assertion (speech act) that might be expressed by that sentence—though in some cases such a reification might have to be rejected, if this is what Russell's "new contradiction" about the totality of propositions in fact shows us. In alethic modal contexts such an intensional object can be associated with the "class" of possible worlds (or contexts) in which the assertion in question would be true.

The main point is that in either Frege's framework or that of conceptual realism, the entities denoted by nominalized predicates and propositional forms (if anything is denoted at all) need not be the same as the entities

[14] See, e.g., Appendix II, p. 128, of Frege 1964.

that the predicates and propositional forms otherwise stand for in their role in predication and assertion.[15]

2. Russell's "New Contradiction"

Let us now turn to Russell's "new contradiction" in Appendix B of PoM. On page 527 of the Appendix, Russell writes:

> If *m* be a class of propositions, the proposition "every *m* is true" may or may not be itself an *m*. But there is a one-one relation of this proposition to *m*: if *n* be different from *m*, "every *n* is true" is not the same proposition as "every *m* is true." Consider now the whole class of propositions of the form "every *m* is true," and having the property of not being members of their respective *m*'s. Let this class be *w*, and let *p* be the proposition "every *w* is true." If *p* is a *w*, it must possess the defining property of *w*; but this property demands that *p* should not be a *w*. On the other hand, if *p* be not a *w*, then *p* does not possess the defining property of *w*, and therefore is a *w*. Thus the contradiction appears unavoidable.

Russell's assumption that if *n* be different from *m*, then the proposition expressed by "every *n* is true" is not the same as that expressed by "every *m* is true" is not itself provable in λHST* or HST^*_λ, and so must be taken as a premise of Russell's argument. Where "*True*" is a one-place predicate constant defined as follows:

$$True =_{\text{df}} [\lambda x(\exists P)(x = P \wedge P)],$$

i.e., where *True* stands for the concept of being a proposition that is the case, we can formulate (the contrapositive of) Russell's assumption as follows:[16]

$$(\forall F)(\forall G)(F \sqsubseteq [\lambda x(\exists P)(x = P)] \wedge G \sqsubseteq [\lambda x(\exists P)(x = P)] \wedge$$
$$[(\forall y)(G(y) \to True(y))] = [(\forall y)(F(y) \to True(y))] \to F = G).$$

That is, if *F* and *G* are concepts under which (at most) only propositions fall, then the proposition that every *G* is true is the proposition that every *F* is true only if the concepts *F* and *G* are identical.[17]

---

[15] Frege took propositions to be objects and not entities of a separate category represented by propositional variables. In this regard HST^*_λ may not be an appropriate reconstruction of his system. But if we distinguish propositions categorially as intensional entities from the class of possible worlds they are true in, as we do in conceptual realism, then such a reconstruction of Frege's logic may be acceptable after all.

[16] We take $\sqsubseteq$ to represent the subordination of one concept under another, i.e.,

$$F \sqsubseteq H =_{\text{df}} (\forall x)[F(x) \to H(x)].$$

[17] Nothing seems to turn on Russell's use of "class" as opposed to our use of "concept" here. That is, the premise seems to be no more plausible for classes than for concepts.

Let us now turn to the next step in Russell's argument, namely, specification of the concept W of being a proposition of the form $(\forall y)(F(y) \to True(y))$, where F is a concept under which that proposition (or rather its object-correlate) does not fall. Using definition by λ-abstract, we specify W as follows:

$$W =_{df} [\lambda x(\exists P)(\exists F)(x = P \wedge P = [(\forall y)(F(y) \to True(y))] \wedge \neg F(x))].$$

Here we note that W is not specified by a λ-abstract that is h-stratified (because if it were, then there would be an assignment t of natural numbers to the terms and predicates of W such that $t(F) = t(x) + 1$, and yet $t(x) = t(P) \geqslant t(F)$, which is impossible). The concept W, in other words, cannot be specified in λHST*, and hence cannot be used in the remainder of Russell's argument for his "new contradiction."

W can be specified as a concept in HST^*_λ, however, even though we cannot prove that it has an object as its correlate. We can continue with Russell's argument in HST^*_λ, in other words. Accordingly, let P be the proposition that every W is true, i.e., let

$$P =_{df} [(\forall y)(W(y) \to True(y))].$$

Given the above assumption of Russell's, we note first that $\neg W(P)$ follows. For if $W(P)$, then, by $(\exists/\lambda\text{-}Conv^*)$,[18] $(\exists x)[P = x \wedge (\exists P')(\exists F)(x = P' \wedge P' = [(\forall y)(F(y) \to True(y))] \wedge \neg F(x))]$, and hence, by Leibniz's law, (LL^*), $P = P' \wedge \neg F(P)$; but then, by Russell's assumption, $F = W$, and therefore, by (LL^*), $\neg W(P)$, which is impossible by the assumption that $W(P)$. Now from $\neg W(P)$ and $(\exists/\lambda\text{-}Conv)^*$, it follows that

$$\neg(\exists x)[P = x \wedge (\exists P')(\exists F)(x = P' \wedge P' = [(\forall y)(F(y) \to True(y))] \wedge \neg F(x))].$$

But, by $\neg W(P)$ and the definition of P,

$$(\exists P')(\exists F)(P = P' \wedge P' = [(\forall y)(F(y) \to True(y))] \wedge \neg F(P)]),$$

and therefore, on pain of contradiction, $\neg(\exists x)(P = x)$, i.e., the proposition P, as specified above, has no object as its correlate, which in conceptual realism means that the truth conditions determined by P cannot be "objectified."

Russell's argument does not lead to a contradiction in HST^*_λ, in other words, but rather shows that just as not every concept can have an

---

[18] $(\exists/\lambda\text{-Conv}^*)$ is the λ-conversion rule in logics that are free of existential presuppositions for singular terms. In the monadic case it is formulated as:

$$[\lambda x\varphi](a) \leftrightarrow (\exists x)(a = x \wedge \varphi),$$

where x is not free in a.

object-correlate so too not every proposition can have an object-correlate. This may seem an odd result, especially if we think of propositions as intensional objects (the way Frege thought of *Gedanken*). But sentences are not (complex) singular terms in natural language, and categorially distinguishing them in formal logic from the category of singular terms has become a well-entrenched tradition. What this result suggests is that there is more to the tradition than meets the eye.[19]

3. Possible Worlds

Unlike the situation in set theory, we can represent different notions of a possible world in a direct manner in λHST* and HST^*_λ if we extend these systems to modal variants in which $\Box$ is taken as a new logical constant (and, by definition, $\Diamond$ as well). The actual positing of possible worlds goes beyond what can be allowed in conceptualism, we believe, but is justified in the framework of logical realism (whether of an early Russellian or Fregean kind), which we will adopt for what follows. We assume for this kind of framework that all of the axiom schemas for the modal propositional logic S5, the rule of necessitation (applied to theorems), and both directions of the Barcan formula for objectual and predicate quantifiers are added to the axioms and rules for λHST* and HST^*_λ. We refer to the result of these extensions as $\Box\lambda$HST* and $\mathrm{HST}^*_{\lambda\Box}$, respectively. The Barcan formulas for predicate quantifiers are required because the properties, relations, and propositions that are referred to by these quantifiers are taken to be the same through all of the different possible worlds. The Barcan formulas for objectual quantifiers are required, at least for abstract intensional objects (which do not "exist" as concreta in the different possible worlds), because they are also assumed to be the same through different worlds. In addition, because all fiction consists only of intensional content—or so we maintain—purely fictional objects can be taken to be abstract intensional objects;[20] and perhaps all merely possible (as opposed to actual) concrete objects can also be interpreted in this way as well—although that is a thesis we will not attempt to defend here. In any case, we do not think the assumption of *possibilia*, and therefore of both directions of the Barcan formula, is really a problem in logical realism.

---

[19] It should be noted that if an axiom of extensionality were added to HST^*_λ, then, at least as applied to propositions, it would have to be restricted to those propositions that have object-correlates—because then every proposition would be identical with a true or a false proposition that would have an object-correlate.

[20] See Cocchiarella 1996 for such an account of fictitious objects, including impossible fictitious objects (such as the round square in the story *Romeo and Juliet in Flatland*).

One well-known notion of a possible world is that of a proposition P that is both possible, i.e., $\Diamond P$, and maximal in the sense that for each proposition Q, either P implies Q or P implies $\neg Q$ (in the sense of necessary material implication), i.e., either $\Box(P \to Q)$ or $\Box(P \to \neg Q)$.[21] Where P is a possible world in this sense, we read "$\Box(P \to Q)$" as "Q is true in P". Note that because this notion can be described by an h-stratified formula, it follows by (CP^*_λ) that the property of being a possible world in this sense "exists" in both $\Box\lambda\mathrm{HST}^*$ and $\mathrm{HST}^*_{\lambda\Box}$. This means that we can explicitly define this notion in terms of a λ-abstract:

$$Poss\text{-}Wld_1 =_{\mathrm{df}} [\lambda x(\exists P)(x = P \wedge \Diamond P \wedge (\forall Q)[\Box(P \to Q) \wedge \Box(P \to \neg Q)])].$$

Of course, that there are possible worlds in this sense is not provable in these systems, unless such an axiom is added to that effect, e.g.,

$$(\exists \mathrm{Wld}_1) \quad \Box(\exists P)(Poss\text{-}Wld_1(P) \wedge P),$$

which says that some possible world (in the sense in question) obtains in every possible world (in the sense of $\Box$), and in particular some possible world now obtains, i.e., $(\exists P)(Poss\text{-}Wld_1(P) \wedge P)$, which we can refer to as "the actual world."

A criterion of adequacy for this notion of a possible world is that it yield the type of results we find in the set-theoretic semantics for modal logic. One such result, assuming $(\exists Wld_1)$, is that a proposition is true, i.e. now obtains, if it is true in the actual world, i.e.,

$$(\exists \mathrm{Wld}_1) \quad \vdash Q \leftrightarrow (\exists P)[Poss\text{-}Wld_1(P) \wedge P \wedge \Box(P \to Q)],$$

(where "$\vdash$" covers provability in both $\Box\lambda\mathrm{HST}^*$ and $\mathrm{HST}^*_{\lambda\Box}$). From this another appropriate result follows; namely, that a proposition Q is possible, i.e., $\Diamond Q$, if it true in some possible world,

$$(\exists \mathrm{Wld}_1) \quad \vdash \Diamond Q \leftrightarrow (\exists P)[Poss\text{-}Wld_1(P) \wedge \Box(P \to Q)].$$

Finally, another appropriate consequence is that if Q and Q' are true in all the same possible worlds, then they are necessarily equivalent; and, conversely, if they are necessarily equivalent, then they are true in all the same possible worlds:

$$(\exists \mathrm{Wld}_1) \quad \vdash (\forall P)(Poss\text{-}Wld_1(P) \to [\Box(P \to Q) \leftrightarrow \Box(P \to Q')]) \leftrightarrow \Box(Q \leftrightarrow Q').$$

Of course, it does not follow that propositions are identical (or have the

[21] See Prior and Fine 1977 for a discussion of this approach to possible worlds.

same object-correlate) if they are true in all the same possible worlds; but if we were to assume an axiom schema of intensionality,

$$(\Box\text{Ext*}) \quad \Box(\forall x_1) \ldots (\forall x_n)[\varphi \leftrightarrow \chi] \rightarrow [\lambda x_1 \ldots x_n \varphi] = [\lambda x_1 \ldots x_n \chi],$$

for all natural numbers n, then such a result would follow.

Another notion of a possible world is that of a property in the sense of one of the "ways things might have been." David Lewis claimed that possible worlds are ways things might have been, but for Lewis the "ways that things might have been" are concrete objects, and not properties.[22] Robert Stalnaker noted, however, that "*the way things are* is a property or state of the world, not the world itself," as Lewis would have it.[23] Stalnaker is an actualist, however, and the idea that possible worlds are properties is his way of affirming the "existence" of possible worlds as uninstantiated properties—except, that is, for the actual world. Presumably, the possible instantiation of such properties would all be concrete objects, and not, e.g., the propositions that are true in those worlds. In our present framework, however, the "ways things might have been" are indeed properties, but they are properties of all and only the propositions that are true in the world in question.

A property of all and only the propositions that are true in a world could not be identified with that world if its extension was different in different possible worlds. What is needed is a property that has the same extension in every possible world, or what might be called a "rigid" property. In general, for all natural numbers n, the statement that an n-ary relation (property if $n=1$, and proposition if $n=0$) is rigid in this sense can be abbreviated as follows:[24]

$$Rigid_n(F^n) : (\forall x_1) \ldots (\forall x_n)[\Box F(x_1, \ldots, x_n) \vee \Box \neg F(x_1, \ldots, x_n)].$$

The type of possible world that is now under consideration is that of a rigid property (or "class") that holds in some possible world (in the sense of $\Diamond$) of all and only the propositions that are true in that world. This notion can be specified by an h-stratified formula, which means that the property of being a possible world in this sense can be λ-defined as follows:

$$Poss\text{-}Wld_2 =_{\text{df}} [\lambda x(\exists G)(x = G \wedge Rigid_1(G) \wedge \Diamond(\forall y)[G(y) \leftrightarrow True(y)])].$$

R. M. Adams describes something like this notion, which he calls a

[22] See Lewis 1973, p. 84. The relevant text is reprinted in Loux 1979, p. 182.

[23] Stalnaker 1976, p. 228.

[24] Rigidity can be λ-defined as a predicate in $\Box\lambda$HST*. In $\text{HST}^*_{\lambda\Box}$, however, it must be construed only as an abbreviation in the principle of rigidity described below.

world-story, in his account of possible worlds.[25] But for Adams, "the development of a satisfactory logic of propositions (or of intensions generally) is also beset by formal problems and threats of paradox," apparently because, according to Adams, such a logic "seems to imply that there are consistent sets composed of one member of every pair of mutually contradictory propositions."[26] Adams, however, is thinking of a framework like ZF set theory, and he does not seem to be aware of such an alternative as NFU, or of such intensional logics as $\Box\lambda$HST* and $\mathrm{HST}^*_{\lambda\Box}$. There is no threat of paradox in NFU or these intensional logics. Of course, as with our first notion, that there are possible worlds in this sense is not provable in $\Box\lambda$HST* or $\mathrm{HST}^*_{\lambda\Box}$, unless we add an assumption to that effect. One such assumption is the following, which says that there is such a possible world that holds (in any possible world in the sense of $\Box$) of all and only the propositions that are true (in that world):

$$(\exists \mathrm{Wld}_2) \quad \Box(\exists G)(\textit{Poss-Wld}_2(G) \wedge (\forall y)[G(y) \leftrightarrow \textit{True}(y)]).$$

It is noteworthy that $(\exists\mathrm{Wld}_2)$ is derivable in both of our modal systems $\Box\lambda$HST* and $\mathrm{HST}^*_{\lambda\Box}$ from what might be called a *principle of rigidity*, (PR), to the effect that every *n*-ary relation F is co-extensive (in any possible world in the sense of $\Box$) with a rigid relation, and therefore with a rigid relation that can be taken as the extension of F (in that world):

$$\Box(\forall F^n)(\exists G^n)(\textit{Rigid}_n(G) \wedge (\forall x_1)\ldots(\forall x_n)[F(x_1,\ldots,x_n) \leftrightarrow G(x_1,\ldots,x_n)]).$$

The idea of representing extensions in this way was first suggested by Richard Montague, and a type-theoretical version was utilized as a principle of extensional comprehension by Dan Gallin in his development of Montague's intensional logic.[27] That $(\exists\mathrm{Wld}_2)$ is derivable from (PR) follows from the fact that *True* represents a property in these systems; that is,

$$(\mathrm{PR}) \quad \vdash(\exists G)(\textit{Rigid}_2(G) \wedge (\forall y)[G(y) \leftrightarrow \textit{True}(y)]),$$

and therefore, by the rule of necessitation and obvious theses of modal logic, $(\exists\mathrm{Wld}_2)$ is derivable from (PR).

It is also noteworthy that $(\exists\mathrm{Wld}_1)$ is also derivable from (PR) in $\Box\lambda$HST* and $\mathrm{HST}^*_{\lambda\Box}$.[28] That is, in the intensional logics $\Box\lambda$HST*+(PR) and $\mathrm{HST}^*_{\lambda\Box}$+(PR), we can prove the existence of possible worlds in both of the senses in question. Thus, not only are Russell's paradoxes and the related modal paradoxes resolvable in these intensional logics, but, in

[25] See Adams 1974, p. 204.
[26] Ibid., p. 207.
[27] See Montague 1974, p. 132, and Gallin 1975, p. 77.
[28] Because the proof involves a number of steps, we will forego going into the details here.

addition, different notions of a possible world can be consistently developed and proved to exist in them as well.

References

Adams, R. M. 1974. Theories of actuality. *Noûs*, vol. 8, pp. 211–231; reprinted in Loux 1979.

Cocchiarella, N. B. 1985. Frege's double-correlation thesis and Quine's set theories NF and ML. *Journal of Philosophical Logic*, vol. 4, pp. 1–39.

Cocchiarella, N. B. 1986. *Logical Investigations of Predication Theory and the Problem of Universals*. Bibliopolis Press, Naples, 1985.

Cocchiarella, N. B. 1992. Cantor's Power-Set Theorem Versus Frege's Double-Correlation Thesis, *History and Philosophy of Logic*, vol. 13, 179–201.

Cocchiarella, N. B. 1996. Conceptual realism as a formal ontology. In R. Poli and P. Simons (eds.), *Formal Ontology,* pp. 27–60. Kluwer Academic Press, Dordrecht.

Frege, G. 1964. *The Basic Laws of Arithmetic*, translated and edited by M. Furth, University of California Press, Berkeley and Los Angeles.

Holmes, R. 1999. *Elementary Set Theory with a Universal Set.* Cahiers du Centre de Logique, Bruylant-Academia, Louvain-la-Neuve, Belgium.

Grim, P. 1991. *The Incomplete Universe.* MIT Press, Cambridge, MA.

Jubien, M. 1988. Problems with possible worlds. In David F. Austin (ed.) *Philosophical Analysis, A Defence by Example.* Kluwer Academic publishers, Dordrecht.

Jensen, R. 1968. On the consistency of a slight (?) modification of Quine's *New Foundations*. *Synthese*, vol. 19, pp. 250–263.

Lewis, D. 1973. *Counterfactuals.* Harvard University Press, Cambridge, MA.

Loux, M. J. 1979. *The Possible and the Actual. Readings in the Metaphysics of Modality*. Cornell University Press, Ithaca.

Oksanen, M. 1979. The Russell-Kaplan paradox and other modal paradoxes; a new solution. *Nordic Journal of Philosophical Logic*, vol. 4, no. 1, pp. 73–93.

Prior, A. and Fine, K. 1977. *Worlds, Selves and Times.* Duckworth Press, London.

Russell, B. 1937. *The Principles of Mathematics*, 2nd edition. W. W. Norton & Co., N.Y.

Stalnaker, R. C. 1976. Possible worlds, *Noûs*, vol. 10, pp. 65–75; reprinted in Loux 1979.

Department of Philosophy,
Indiana University, USA
e-mail: cocchiar@indiana.edu