Grigori Mints

THORALF SKOLEM AND THE EPSILON SUBSTITUTION METHOD FOR PREDICATE LOGIC*

1. Introduction

Skolem's contributions to mathematical logic are fundamental and far-reaching. A good survey by Hao Wang (1970) is presented in Fenstad's collection of Skolem's works (1970). Even Skolemization, i.e. the replacement of quantifiers by new constant and function symbols, is too extensive for one talk. Cf. my survey (Mints 1990) of proof-theoretic aspects and more recent work (Shankar 1992) concerning dynamic Skolemization in automated deduction.

Here I relate some of Skolem's work (Skolem 1929, §5) to the ϵ-substitution method, which was the center of interest of Hilbert's school in 1920–1930. Hao Wang (1970, p. 26) says: "It is hard to evaluate the importance of the proofs and comments in this section". I show that constructions in Skolem 1929, §5 can be seen as instances of the ϵ-substitution method and the proof of ϵ-theorems. They provide speed-up estimates. I review this material in section 2 and then describe some of the later developments taking models (solutions in Skolem's terminology) into account.

2. Quantifier-Free Expansions of Formulas and ϵ-theorems

In this section $U, V, \ldots$ denote arbitrary quantifier-free predicate formulas with function symbols but without equality. It is well known that every predicate formula can be put into this form by Skolemization. For example,

$$\exists w \forall x \exists y \forall z \exists u U(w,x,y,z,u) \mapsto \; U(c,x,f(x),z,g(x,z))$$

*Delivered as the 1994 Skolem Lecture at the University of Oslo.

Nordic Journal of Philosophical Logic, Vol. 1, No. 2, pp. 133–146.

Recall some constructions (introduced by Skolem) in the neighborhood of the Herbrand theorem.

The *universe* M_k of level k for a formula U is the set of all terms of depth k constructed from the function symbols and constants in U, and a constant 0.

The *expansion* U^k of order k of U is the finite conjunction of all instances of U obtained when free variables range over all elements of M_k.

The *Solution* of level k for U is any model L (truth-value assignment to atomic subformulas of U^k) verifying U^k:

$$L \models U^k$$

The Herbrand theorem is equivalent to a Skolem-type formulation: the universal closure $\forall U$ is inconsistent iff U^k is inconsistent for some k, i.e. there is no solution of level k.

Skolem (1929, §5) considers (following Hilbert) the *Vertreteraxiom*

$$\forall x \exists y \forall z (\neg V(x, y) \vee V(x, z)) \tag{1}$$

which is a prenex normal form of the law of excluded middle

$$\forall x (\exists y \neg V(x, y) \vee \forall z V(x, z)) \tag{2}$$

The main result of this section is summarized in its title as the consistency of adding axioms (1). This may seem strange, since excluded middle is obviously true in any model, and hence adding it simply does not change the truth-value of a sentence. In fact, Skolem produces bounds for the levels of solutions and inconsistent expansions U^k.

First he considers the case when the quantifier $\forall x$ in (1) is redundant, in which case adding (1) is the same as adding

$$Z' \equiv \neg V(a) \vee V(z)$$

for a new constant a and a new variable z. Skolem proves the following statement.

LEMMA 1. *For every solution L_{2k} of level $2k$ for U there is a solution L'_k of level k for $U \& Z'$:*

$$L'_k \models U \& Z'$$

Proof. This is simple. Consider the restriction L_k of L_{2k} to the universe M_k of level k. If there is a truth-value assignment to predicate symbols in $Z' - U$ over M_k which verifies all instances of $V(z)$, take such an assignment W_k and set $a := 0$, i.e. define

$$L'_k(\rho) = L_k W_k(\rho[a := 0])$$

for any expression ρ.

If for every W_k there is a $b \in M_k$ such that $L_k W_k \models \neg V(b)$, then set $a := b$, i.e. fix W_{2k} arbitrarily and define

$$L'_k(\rho) = L_{2k} W_{2k}(\rho[a := b])$$

□

Since there is only a finite number of potential solutions, this proof is constructive (as Skolem stresses), as is the following corollary.

COROLLARY 1. *If $(U \& Z')^k$ is inconsistent, then U^{2k} is inconsistent.*

Skolem then iterates Lemma 1 to obtain a similar result for (1). After this he presents a kind of sketch for extension of this construction to the law of excluded middle with formulas V of arbitrary quantifier complexity.

In view of the close connection between cuts on a formula C and adding of the axiom $\neg C \vee C$ to a cut-free formulation (Kreisel and Takeuti 1974), one can consider this to be a cut-elimination result.

However, the proof of Lemma 1 also reminds one of Hilbert's *Ansatz* (approach) to the proof of the ϵ-theorem. Indeed, replacing V with $\neg V$ in $\neg V[a] \vee V[z]$, substituting $a := \epsilon x V[x]$ and instantiating variable z one obtains the ϵ-axiom (critical formula)

$$V[t] \to V[\epsilon x V[x]] \tag{3}$$

where t may contain $\epsilon x V[x]: \ t = t'[\epsilon x V[x]]$. The transformation used in the proof of Lemma 1 is highly reminiscent of Hilbert's Ansatz for elimination of the critical formula (3) from a proof: substitute $\epsilon x V[x]$ first by 0 and then by $t'[0]$. Then (3) is replaced by a disjunction

$$(V[t'[0]] \to V[0]) \vee (V[t'[t'[0]]] \to V[t'[0]])$$

which is a tautology. The possibility of eliminating all critical formulas (and hence all axioms (1)) from the proofs of free variable formulas in predicate logic (the first ϵ-theorem) was proved in Hilbert and Bernays 1939. It is pointed out there (end of vol. 2, section 2.3.d) that this proof can be traced to W. Ackermann.

3. ϵ-substitution Method for Interpreted Theories and its Transfer to Predicate Logic

The epsilon substitution method is based on the language introduced by Hilbert and Bernays (1939) (and later by Bourbaki (1958)). The main non-Boolean construction of this language is an ϵ-term $\epsilon x F[x]$, read as "an x satisfying the condition $F[x]$". In number-theoretic contexts it is often interpreted as "the least x satisfying $F[x]$". Existential and universal quantifiers become explicitly definable by

$$\exists x F[x] = F[\epsilon x F[x]]; \qquad \forall x F[x] = F[\epsilon x \neg F[x]] \tag{4}$$

The main axioms of the corresponding formalism are *critical formulas*

$$F[t] \to F[\epsilon x F[x]] \tag{5}$$

Hilbert's approach to transforming arbitrary (non-finitistic) number-theoretic proofs into finitistic (combinatorial) proofs by means of the substitution method is described in Hilbert and Bernays 1939. Cf. also the short and lively presentation by Hermann Weyl (1944). The approach was sketched above. In more detail it is as follows.

Take all critical formulas (5) occurring in a given proof P. There is only a finite number of them, so one always deals with a finite system E of critical formulas. Consider any substitution S of numerals for constant ϵ-terms. If all critical formulas (5) are true under S, it is called a solving substitution for the system E. Hilbert proposed a specific plan for finding a solving substitution by a series of successive approximations, described below. If it succeeds and if the last formula of the proof P, i.e. the formula proved by P, is a constant combinatorial identity such as $1+2+\ldots+10 = 55$, replacing all free variables with any numerals and then each ϵ-term t by $S(t)$ immediately yields a variable-free (finitistic, combinatorial) proof of the same identity. Moreover, it was noted by Ackermann (1940) and stressed later by Kreisel that the same device allows one to extract the numerical content of existential proofs, i.e. proofs of existential formulas $\exists x F[x]$ with combinatorial (free variable) $F[x]$. Indeed, $\exists x F[x]$ is translated as $F[\epsilon x F[x]]$. If S is a solving substitution for the proof P of such a formula, and $N = S(\epsilon x F[x])$ then $S(P)$ is a proof of $F[N]$. So N is a numerical realization of the existential quantifier in $\exists x F[x]$.

Hilbert's suggestion for finding a solving substitution by a successive approximation method is based on the following idea of generating substitutions of numerals for closed ϵ-terms. The initial approximation S_0 is identically 0: every ϵ-term is assigned the value 0. At every stage only a finite number of ϵ-terms are assigned non-zero values. If approximations $S_0, \ldots, S_i$ are already generated, and S_i is not yet a solving substitution, then S_{i+1} is found by correcting the value of some $\epsilon x F[x]$.

The problem stated by Hilbert was to prove termination of the sequence $S_0, S_1, S_2, \ldots$ after a finite number of steps for any system of critical formulas (5).

After Gentzen's proof (1936) of cut-elimination and consequently of consistency for arithmetic (with respect to closed equations) revealed the role of the ordinal ϵ_0, Ackermann (1940) was able to find a final formulation and to give a termination proof for full first-order arithmetic (pure number theory). This proof was preceded by a termination proof (also due to Ackermann) for a version of the substitution method for the first-order predicate logic with equality and extensionality, which is presented in Hilbert and Bernays 1939. This method proceeds 'from above' similarly to cut-elimination.

I present here a definition of epsilon substitution 'from below' for predicate logic. Extensions by equality and extensionality, as well as termination proof for the substitution process and derivation of corresponding Herbrand-type theorems, are done in Mints 1995. In fact, I there prove strong termination: every (not only some special) sequence of consecutive ϵ-substitutions terminates in a solution.

Note that the substitution method for predicate logic as described in Hilbert and Bernays 1939 is not strongly terminating: the attempt to eliminate terms of lower degree first can lead to a loop.

4. Predicate Calculus. The System $PC\epsilon$

Let L be a language of the first-order predicate calculus (without equality) with free individual variables denoted by a, b, $\ldots$ and bound individual variables by x, y, z, $\ldots$. Then L_ϵ will denote the extension of L by Hilbert's ϵ-symbol. More precisely, terms and formulas of L_ϵ are defined simultaneously in the standard way including the clause: if $F[a]$ is a formula then $\epsilon x F[x]$ is a term.

Since quantifiers are definable in terms of the ϵ-symbol:

$$\exists x F[x] = F[\epsilon x F[x]]; \qquad \forall x F[x] = F[\epsilon x \neg F[x]]$$

I am interested mainly in quantifier-free formulas of L and L_ϵ.

An *expression* is a term or a formula. A *quasiexpression* (*quasiterm, quasiformula*), or briefly *qexpression* (*qterm, qformula*), is obtained from an expression (term, formula) by replacing some occurrences of free variables with bound ones.

By a *subterm* of an expression I mean an *occurrence* of a term in it. A subterm t of a term or formula e is *proper* if it does not coincide with e. An *exterior proper* subterm of e is a proper subterm of e which is not a proper subterm of any proper subterm of e. By a *subqterm* of a qexpression I mean an *occurrence* of a qterm in it.

I use metavariables s, t, ... for qterms, F, G, ... for qformulas, e, g, ... for qexpressions.

The *matrix* of a term is obtained by replacing all its exterior proper ϵ-subterms with new distinct free variables.

Derivable objects of the system $PC\epsilon$ are formulas.

Axioms are substitution instances of propositional tautologies and critical formulas

$$F[t] \to F[\epsilon x F[x]] \tag{6}$$

The only *rule of inference* is modus ponens.

The term $\epsilon x F$ in a critical formula is the *main term*, and the term t is the *side term*.

Recall (Hilbert and Bernays 1939) that $PC\epsilon$ is equivalent to first-order classical logic.

In the following we are interested in *term models* $\mathcal{M}$ of the language L, where the universe consists of all terms of L (i.e. ϵ-free terms), and hence they are treated as constants. Functional symbols of $\mathcal{M}$ are defined "formally":

$$\mathcal{M}(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$$

Terms of L will be denoted by $c, c_1, \ldots$.

5. Epsilon Substitution Method for PCϵ

Assume that a finite system E of critical formulas is fixed.

An expression is *canonical* iff it does not contain proper ϵ-subterms.

The *rank* $rk(e)$ of an expression e is defined in the following well-known way: The rank of an ϵ-free term is 0. $rk(f(t_1, \ldots, t_k)) = \max(rk(t_1), \ldots, rk(t_k))$. $rk(\epsilon x F[x]) = \max(rk(\epsilon y G) \mid F[a]$ contains subterm $\epsilon y G$ containing $a) + 1$. The rank of any other expression is the maximum rank of ϵ-terms occurring in it. The rank of a critical formula (6) is the rank of its main term, i.e. $rk(\epsilon x F[x])$.

Lemma 2. *If s and t are terms and $e[a]$ is an ϵ-term then*

$$rk(e[s]) = rk(e[t]) = rk(e[a])$$

Proof. By induction on length of ϵ-terms. □

LEMMA 3. *For any expression* $e[a]$, *for any term* t, *for any* ϵ*-free term* c

$$rk(e[c]) \leq rk(e[t])$$

Proof. Immediate from the previous Lemma. □

The *degree* $d(e)$ of a term e is defined as the maximum nesting of its ϵ-subterms: $d(e) = 0$ for any ϵ-free expression e. $d(\epsilon xA) = 1$, if ϵxA has no proper ϵ-subterms. $d(f(t_1, \ldots, t_n)) = \max(d(t_1), \ldots, d(t_n))$, if f is a function constant. $d(\epsilon xA[x, t_1, \ldots, t_n]) = \max(d(t_1), \ldots, d(t_n)) + 1$, where $t_1, \ldots, t_n$ are all outermost proper ϵ-subterms of $\epsilon xA[x, t_1, \ldots, t_n]$.

5.1. Epsilon Substitution

The definition presented here is intended to be as similar as possible to Ackermann's (Ackermann 1940, Hilbert and Bernays 1939) definition for first-order arithmetic. A less essential difference concerns syntactic form: Ackermann uses finite functions as interpretations of ϵ-matrices, and I (following Mints and Tupailo 1993) use finite sequences of equations to represent graphs of such functions. A much more important difference is the use of the standard model for arithmetic in Ackermann's construction: the values of all constant terms become computable as soon as an ϵ-substitution is fixed. The same applies to the truth-values of constant quantifier-free formulas (containing no ϵ).

For predicate logic the same situation is achieved if some term model of the language L is fixed. This motivates the definition of the ϵ-substitution and reduction sequence of ϵ-substitutions given below.

DEFINITION 1. Epsilon substitution *(*ϵ-substitution *or simply* substitution*) is a finite list of equations*

$$\epsilon xF_1 = c_1, \ldots, \epsilon xF_k = c_k, \tag{7}$$

where ϵxF_i *are distinct canonical* ϵ*-terms, and* c_i *are* ϵ*-free terms. The terms* ϵxF_i *are the main terms of the corresponding equations;* c_i *are their values.*

5.2. Computations under a given ϵ*-substitution*

ϵ-substitution (7) is finite, but it will be extended to all ϵ-terms in a standard way, i.e. by assigning a default value to all ϵ-terms outside the domain of (7). We fix a constant (or a free variable if there are no constants) of the language L as a default value, and denote it by 0.

Let S be an epsilon substitution. Below we define *computation step* $e \overset{S}{\rightharpoonup} g$ (abbreviated $e \rightharpoonup g$). The *computation relation* $\overset{S}{\longrightarrow}$ (abbreviated $\longrightarrow$) is defined as the reflexive transitive closure of $\overset{S}{\rightharpoonup}$.

DEFINITION 2. COMPUTATION STEP. (i) *$\epsilon xF \rightharpoonup c$ if $\epsilon xF = c$ is in S.* (ii) *$\epsilon xF \rightharpoonup 0$ if $\epsilon xF = c$ is not in S for any c, but ϵxF is a canonical ϵ-term.* (iii) *$e[\mathbf{u}] \rightharpoonup e[\mathbf{n}]$ if $e[\mathbf{u}]$ is an expression containing proper ϵ-subterms, $\mathbf{u}$ are all its deepest ϵ-subterms, $\mathbf{u} \longrightarrow \mathbf{n}$ and $\mathbf{n}$ is a list of ϵ-free terms.*

DEFINITION 3. *The* S-value *$S(e)$ of an expression e is any ϵ-free expression such that $e \overset{S}{\longrightarrow} S(e)$. The* S-canonical form *of an ϵ-term e is a canonical ϵ-term $CanS(e)$ such that $e \overset{S}{\longrightarrow} CanS(e)$.*

Note 1. I defined computations from inside. The S-value of an expression is an expression of the same sort. For example, the S-value of an ϵ-term is an ϵ-free term, the S-value of a quantifier-free formula is an ϵ-free quantifier-free formula, i.e. quantifier-free formula of L.

LEMMA 4. EXISTENCE AND UNIQUENESS OF S-VALUES. *For any ϵ-substitution S, expression e and ϵ-term v the values $S(e)$ and $CanS(v)$ exist and are unique.*

Proof. Uniqueness follows from the fact that at most one computation step is applicable to an expression e. Indeed, if e contains proper ϵ-subterms then the only possible computation step is replacement of all deepest ϵ-subterms of e by their S-values. If e is a canonical ϵ-term then it is computed in one step to an ϵ-free term determined by S.

Existence is proved by induction on the number of ϵ-symbols. □

LEMMA 5. *For any ϵ-substitution S and for any expression $e[u]$, if u is an exterior proper ϵ-subterm of e then*

$$S(e[u]) = S(e[S(u)])$$

Proof. By easy induction on e. □

LEMMA 6. CHURCH-ROSSER PROPERTY. *For any S and $e[t]$, if t is a subterm of $e[t]$ then*

$$S(e[t]) = S(e[S(t)])$$

Proof. By induction on length of $e[t]$. If $e[t]$ is t then the assertion is evident. Otherwise by the previous Lemma it is sufficient to prove the assertion for the exterior proper ϵ-subterm of $e[t]$ containing t. The latter has lesser length and it is possible to apply the induction hypothesis to it. □

LEMMA 7. *If S is an ϵ-substitution and e is an expression, then the rank of any ϵ-term used in the computation of $CanS(e)$ and $S(e)$ does not exceed the maximum rank of ϵ-terms occurring in e.*

Proof. Computation step does not increase rank. □

5.3. Reduction sequence of ϵ-substitutions

As noted in 5.1, a model is needed to determine truth-values of formulas. Assume that a term model $\mathcal{M}$ for the language L is fixed. Each term or a formula e of the language L has a value in $\mathcal{M}$, which is denoted by $\mathcal{M}(e)$. Indeed, if e does not contain free variables, its value in $\mathcal{M}$ is computed in the standard way. If e contains free variables, they are elements of the universe of $\mathcal{M}$, and e can still be treated as a closed term or formula. If e is a term, then $\mathcal{M}(e) = e$ is a term, and if e is a formula, then $\mathcal{M}(e)$ is a truth-value $\top$ or $\perp$. Combining this with the previous section, we define the value under S and $\mathcal{M}$ ($S\mathcal{M}$-value for short) by

$$S\mathcal{M}(e) = \mathcal{M}(S(e)) \tag{8}$$

for any term or formula e of the language L_ϵ.

Note 2. If e is a term of the language L_ϵ, then $S\mathcal{M}(e)$ is a term of L; if e is a formula of L_ϵ, then $S\mathcal{M}(e)$ is a truth-value $\top$ or $\perp$. In particular, each critical formula (6) has a truth-value under $S\mathcal{M}$.

Note 3. The canonical form of an ϵ-term is not redefined in connection with $\mathcal{M}$.

DEFINITION 4. *An ϵ-substitution S is correct under $\mathcal{M}$ iff for every equation $\epsilon x F[x] = c$ occurring in S one has $S\mathcal{M}(F[c]) = \top$.*

Let E be a finite system $\{Cr_I \mid I = 1, \dots N\}$ of critical formulas, S an ϵ-substitution (7) and $\mathcal{M}$ a term model of L. Suppose that $S\mathcal{M}(E) = \perp$, i.e. one of the critical formulas is false under $S\mathcal{M}$. Choose a false critical formula

$$G[t, \mathbf{u}] \to G[\epsilon x G[x, \mathbf{u}], \mathbf{u}] \tag{9}$$

with the outermost proper subterms $\mathbf{u}$. Then S' is obtained from S as follows:

(i) The equation

$$\epsilon x G[x, S(\mathbf{u})] = S(t) \tag{10}$$

is added. (ii) If an equation $\epsilon x G[x, S(\mathbf{u})] = v$ for some v occurs in S, then it is deleted (in fact this never happens). (iii) All equations with rank $> rk(\epsilon x G[x, \mathbf{u}])$ are deleted.

This operation of transforming S into S' is called a *reduction* of S or an *ϵ-step* (for the critical formula (9)). We say that (9) is *reduced* at this step. The *rank* of a reduction is the rank of the equation which is added in it.

Note. Since the critical formula (9) was false under $S\mathcal{M}$, one has

$$S\mathcal{M}(G[\epsilon x G[x,\mathbf{u}],\mathbf{u}]) = S\mathcal{M}(G[0,S(\mathbf{u})]) = \bot$$

and

$$S\mathcal{M}(G[t,\mathbf{u}]) = S\mathcal{M}(G[S(t),S(\mathbf{u})]) = \top \tag{11}$$

A *reduction sequence* of ϵ-substitutions is any (finite or infinite) sequence of reductions:

$$\emptyset = S, \quad S', \quad S'', \quad \ldots$$

A *standard reduction sequence* (considered by Ackermann (Hilbert and Bernays 1939)) is a special reduction sequence with the following rule of choosing a false critical formula for the reduction: it has to be the first critical formula of minimal rank.

Note that a reduction sequence terminates only if a *solution* is reached: all formulas in E are true.

The problem of *termination* of the ϵ-substitution method is the problem of termination of at least one reduction sequence, for instance the standard one. The problem of *strong termination* is the problem of termination of all reduction sequences. Termination and strong termination can be thought of as analogues of normalization and strong normalization. In the next section I prove strong termination, i.e. termination of all possible reduction sequences.

Lemma 8. Ackermann's Lemma. *In all reduction sequences all ϵ-substitutions are correct.*

Proof. The initial empty substitution is obviously correct. Let S be correct, $S\mathcal{M}(E) = \bot$, false critical formula (9) with $rk(\epsilon x G[x,\mathbf{u}]) = r$ be chosen for reduction and equation (10) be added, so that (11) holds. Note that all ϵ-terms occurring in the next ϵ-substitution S' are canonical ϵ-terms of ranks $\leq r$. Consider any equation $\epsilon x A[x] = c$ which is inherited by S' from S. We have $rk(A[c]) < r$ for any ϵ-free term c. Since parts of S and S' of ranks $< r$ coincide, by the Church-Rosser property (Lemma 6) we have

$$S(A[c]) = S'(A[c]) \tag{12}$$

and hence $S\mathcal{M}(A[c]) = S'\mathcal{M}(A[c]) = \top$. For the equation (10) added in the ϵ-step the requirement of correctness is satisfied by (11, 12). □

COROLLARY 2. *Case (ii) of the definition of ϵ-step never occurs in a reduction sequence: non-default values can be simply deleted; they are never changed to other non-default values.*

Proof. Let an ϵ-substitution S occur in a reduction sequence. Then S is correct. If $Can(e) = g$ occurs in S, then $S(Cr)$ is true since its conclusion is true. □

THEOREM 3. *All reduction sequences for E starting with $\emptyset$ terminate.*

Proof. Cf. Mints 1995. □

6. The Tree of Reduction Sequences

The definition of a reduction sequence in section 5.3 and the termination proof in section 3 work without change if $\mathcal{M}$ is only a partial model, provided all necessary computations can be performed.

DEFINITION 5. *A* diagram *is any (finite or infinite) sequence of literals of the language L containing no pair of complementary literals.* Recall that a *literal* is an atomic formula or a negation of an atomic formula.

DEFINITION 6. *Diagram $\mathcal{M}$ is* suitable *for a system E of critical formulas and ϵ-substitution S, if for any atomic subformula e of E or of a formula $F[c]$ for $\epsilon x F[x] = c$ in S the value $\mathcal{M}(S(e))$ is determined, that is*

$$S(e) \in \mathcal{M} \qquad \text{or} \qquad (\neg S(e)) \in \mathcal{M}$$

For a suitable diagram *the value* $S\mathcal{M}(e)$ is defined as before.

Note that the computations needed to verify *correctness* of a substitution still go through. A *reduction sequence* for E with respect to a diagram $\mathcal{M}$ is defined as before for models (section 3.3), with the additional possibility of a deadlock, if $\mathcal{M}$ is not suitable for E and S_i for some i.

DEFINITION 7. *A diagram $\mathcal{M}$ is* suitable for a reduction sequence *(for a system E) if the deadlock never occurs.*

THEOREM 4. *If $\mathcal{M}$ is suitable for a reduction sequence, then this reduction sequence is finite.*

Proof. The same as for Theorem 1. □

Next I describe a primitive recursive finitely branching tree of finite ϵ-substitutions for a given system E and finite diagrams, and prove the tree to be well-founded. This will imply that every reduction sequence together with every sequence of suitable finite diagrams terminates.

Definition 8. *A suitable extension of a diagram $\mathcal{M}$ with respect to an ϵ-substitution S and a system E of critical formulas is any diagram*

$$\mathcal{M}' \equiv \mathcal{M} \cup \{L_1, \ldots, L_k\}$$

such that (i) *literals $L_1, \ldots, L_k$ are not decided by $\mathcal{M}$, i.e. none is in $\mathcal{M}$ up to negation;* (ii) *$L_1, \ldots, L_k$ occur up to negation in $S(E)$ or in a formula $S(F[c])$ for $\epsilon x F[x] = c$ in S, and $\mathcal{M}'$ is suitable for S.*

The tree $T_\epsilon(E)$ is defined as follows. Its nodes contain pairs $(S, \mathcal{M})$ where S is an ϵ-substitution and $\mathcal{M}$ is a finite diagram. The root of the tree (situated at level 0) contains the pair $(\emptyset, \emptyset)$ with empty components.

Immediate successors of a pair $(S, \mathcal{M})$ situated at an even level contain exactly pairs $(S, \mathcal{M}')$ with the same substitution S and $\mathcal{M}'$ being a suitable extension of $\mathcal{M}$ with respect to S, E. There are exactly 2^k of such extensions.

Immediate successors of a pair $(S, \mathcal{M})$ situated at an odd level contain exactly pairs $(S', \mathcal{M})$ with the same diagram $\mathcal{M}$ and S' being a result of making a reduction for S.

A pair $(S, \mathcal{M})$ is *terminal* if $(S, \mathcal{M})$ is the solution for the system E, that is

$$S\mathcal{M}(E) = \top$$

Note. One can visualize the extension of a pair $(S, \mathcal{M})$ situated at an even level as a series of cuts over literals $L_1, \ldots, L_k$. Hence at each stage of construction of the tree the disjunction over all diagrams $\mathcal{M}_1, \ldots, \mathcal{M}_q$ in the final nodes (leaves) at this stage is a tautology.

Theorem 5. *The tree $T_\epsilon(E)$ is finite.*

Proof. Each path in this tree determines a reduction sequence and a diagram $\mathcal{M}$ suitable for this reduction sequence. By the previous theorem each path is finite, hence the tree is well-founded. Now apply the König Lemma. □

This result implies a version of the Herbrand theorem for ϵ-calculus.

Theorem 6. *For every system E of critical formulas there is a finite set of ϵ-substitutions*

$$S_1, \ldots, S_p \tag{13}$$

such that the disjunction

$$S_1(E) \vee \ldots \vee S_p(E) \tag{14}$$

is a tautology.

Proof. Consider the tree $T_\epsilon(E)$. By the previous theorem it is finite. Let

$$(S_1, \mathcal{M}_1), \ldots, (S_p, \mathcal{M}_p)$$

be all its leaves (final nodes). This means that

$$S_i \mathcal{M}_i(E) \equiv \top \text{ for each } i = 1, \ldots, p$$

that is

$$\mathcal{M}_i \models S_i(E) \text{ for each } i = 1, \ldots, p$$

By the Note above, the disjunction over all $\mathcal{M}_i$ is a tautology. This concludes the proof. □

References

Ackermann, W. 1940. Zur Widerspruchsfreiheit der Zahlentheorie. *Mathematische Annalen*, 117, pp. 162–194.

Bourbaki, N. 1958. *Theorie des ensembles*. Hermann, Paris.

Gentzen, G. 1936. Die Widerspruchsfreiheit der reinen Zahlentheorie. *Math. Ann.*, 112, no. 4, pp. 493–565.

Hilbert, D. and Bernays, P. 1939. *Grundlagen der Mathematik, Bd. 2*. Springer.

Kreisel, G. and Takeuti, G. 1974. Formally self-referential propositions for cut free classical analysis and related systems. *Dissertationes-Math.*, 118.

Mints, G. 1990. Normalization theorems for the intuitionistic systems with choice principles. In P. Petkov (ed.), *Mathematical Logic*, pp. 59–66. Plenum Press, New York.

Mints, G. 1995. An epsilon substitution method for predicate logic. *Zapiski Nauchnyh Seminarov POMI*, 220, pp. 93–122. (Russian.) English translation to be published in *Journal of Russian Mathematics*.

Mints, G. and Tupailo, S. February 1993. Epsilon substitution method for elementary analysis. Report CSLI-93-175, Stanford University.

Shankar, N. 1992. Proof search in the intuitionistic sequent calculus. In D. Kapur (ed.), *Automated Deduction—CADE 11*, vol. 607 of *Springer Lecture Notes*, pp. 522–536. Springer.

Skolem, Thoralf. 1929. Über einige Grundlagenfragen der Mathematik. *Skrifter utgit av Det Norske Videnskaps-Akademi i Oslo I. Matematisk-naturvidenskabelig klasse*, 4, pp. 1–49. Reprinted as 1929a in Skolem 1970.

Skolem, Thoralf. 1970. *Selected Works in Logic,* edited by Jens E. Fenstad. Universitetsforlaget, Oslo.

Wang, Hao. 1970. A survey of Skolem's work in logic. In Skolem 1970, pp. 17–52.

Weyl, H. 1944. David Hilbert and his mathematical work. *Bull. Am. Math. Society*, 50, pp. 612–654.

Department of Philosophy
Stanford University, U.S.A.