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THORALF SKOLEM
PIONEER OF COMPUTATIONAL LOGIC

1. Introduction

In a way, the title is misleading. Thoralf Skolem did not—as far as I know—use a computer or do any serious computations in logic. He was one of a handful of men who founded modern logic. But in his work he always stressed the computational aspects, perhaps more so than the other founders of logic, and is therefore close to the interests of computer scientists. In this paper I examine Skolem's main ideas.

Skolem's major works in logic are gathered in *Selected Works in Logic* (Universitetsforlaget, Oslo 1970). In this paper I use the system of references from the bibliography of Skolem's work there.

2. Vita

Thoralf Albert Skolem was born in Norway in 1887. He graduated in mathematics—with honours—from the University of Oslo (Kristiania) in 1913, where he became docent in 1918. From 1930 to 1938 he was a scholar at the Christian Michelsen's Institute in Bergen, and from 1938 professor of mathematics at the University of Oslo.

He published about 190 papers—most of them small notes. Two-thirds were published in Norwegian journals—but the important ones in logic are available in *Selected Works in Logic*. One-third of his papers are in mathematical logic, the rest in algebra and number theory (especially Diophantine equations). Skolem died in 1963, just as his last three papers were about to be published. He did not have any pupils in logic. In fact, he hardly lectured in logic at all, preferring number theory, by which he considered it easier to get to the mathematical point.

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The main influence of Skolem's work seems to come from himself. There is a permanence in his themes throughout his whole career—quite independent of the fashions of the time. He was considered to be in fashion in the 1930s and 1940s—and perhaps less so in the 1950s and 1960s. Now, with the emphasis there is on the computational aspects of logic, all the reasons are in place as to why he should be in fashion again.

3. Diophantine equations and logic

I believe that Skolem considered Diophantine equations and logic as one subject. Let me indicate a little speculatively how Skolem's thought might have been. The theory of Diophantine equations considers questions about satisfiability of equations like

$$4x^3y^4 + 3y^3 - 13z^5 = 0$$

This is really not far from questions about satisfiability of logical statements. A statement can be seen as a "Diophantine equation" where we have

true	1
false	0
not F	$1 - F$
F and G	$\min(F, G)$
F or G	$\max(F, G)$

In addition, we have to indicate how the quantifiers are to be interpreted. This is of course more complicated. So Skolem's first work was to see how the quantifiers could be interpreted.

4. Introducing new symbols

Skolem had a simple argument that it was only necessary to consider Diophantine equations of degree ≤ 4. Let us start with the Diophantine equation above

$$4x^3y^4 + 3y^3 - 13z^5 = 0$$

First we observe that we can use a system of equations of degree ≤ 2 instead

$$a = x^2$$
$$b = ax$$
$$c = y^2$$
$$d = c^2$$
$$e = cy$$
$$f = z^2$$
$$g = f^2$$
$$h = gz$$
$$4bd + 3e - 13h = 0$$

Then write this as a single equation by using a sum of squares

$$(a - x^2)^2 + (b - ax)^2 + (c - y^2)^2 + (d - c^2)^2 + (e - cy)^2 +$$
$$(f - z^2)^2 + (g - f^2)^2 + (h - gz)^2 + (4bd + 3e - 13h)^2 = 0$$

We get an equation of degree ≤ 4. Skolem used the same trick with satisfiability of logical statements. Consider the following logical statement

$$\forall x \exists y \forall z \exists u . Rxyzu$$

The satisfiability of this statement is the same as the satisfiability of the conjunction of the following statements

$$\forall xyz.(Axyz \leftrightarrow \exists u.Rxyzu)$$

$$\forall xy.(Bxy \leftrightarrow \forall z.Axyz)$$

$$\forall x.(Cx \leftrightarrow \exists y.Bxy)$$

$$\forall x.Cx$$

Here A, B, and C are new relation symbols. The important point is that if we write the conjunction in prenex normal form, we can get a statement with the $\forall$-quantifiers outermost. We say that the statement is in $\forall\exists$-form. This is the Skolem normal form from his 1920a paper.

THEOREM 1. 1920a. *Questions about satisfiability of logical statements can be reduced to satisfiability of* $\forall\exists$*-statements by introducing new relation symbols.*

5. Countable domains suffice

For such statements it is easy to see that it suffices to consider countable or finite domains. This is Skolem's proof of the Skolem-Löwenheim theorem.

Theorem 2. 1920a. *For satisfiability of a countable set of logical statements it suffices to consider domains which are finite or countable.*

Skolem went on and generalized to statements containing countable conjunctions and disjunctions—and also to $\forall\exists$-statements where we may have a countable sequence of $\exists$-quantifiers. These generalizations are of course quite natural as long as we are not confining ourselves to first-order logic. Skolem thought about these problems as some sort of Diophantine equations—and then it is natural.

Skolem did not introduce Skolem functions. For the satisfiability he used what he called a selection principle to satisfy the quantifier combination. There are two possible proofs of the Skolem-Löwenheim theorem. In the first proof one starts with a model and then uses axiom of choice to select a countable submodel. The other proof works directly with a model made out of terms. Skolem used the first proof in 1920a, the other proof in 1922b and in 1928a. This proof does not require the axiom of choice. Only König's lemma is needed.

6. Set theory and Skolem's paradox

The 1922b paper was an invited one for presentation at the 5th Scandinavian Mathematical Congress in Helsinki. Here Skolem made a number of important remarks on set theory. He observed that Zermelo's use of "definite property" was not as precise as it should be. So Skolem replaced the subset axiom of Zermelo with an axiom scheme running over all first-order definable properties. He also introduced the replacement axiom. About the same time, A. Fraenkel also introduced the replacement axiom. Skolem did this much more clearly than Fraenkel, and it must be a historical accident that the axiom system is called the Zermelo-Fraenkel system rather than the Zermelo-Skolem system. Fraenkel was as loose as Zermelo about the definite properties and his version of the replacement axiom does not do much more than the subset axiom. Perhaps the reason is that Skolem was not particularly interested in set theory. Fraenkel, on the other hand, eagerly promoted the field.

It follows from the Skolem-Löwenheim theorem that if set theory has a model at all, it has a countable model. (This requires of course Skolem's explanation of definite property as a first-order axiom scheme.) For Skolem this was a paradox—the only way he could make a sensible system for set theory ended up with unwanted models. Skolem was aware of this in 1915 and reported it to F. Bernstein. There were two consequences. Skolem started to distrust set theory. He saw no way of making the theory clear for himself, and he became even more a constructivist than he had been before. The other consequence was that he saw it perhaps as impossible to avoid unintended models.

Skolem observed that Zermelo's axiom system was satisfied for all sets of finite order. To go beyond that, he introduced the replacement axiom. He also considered the universe of all sets with no infinite descending $\in$-chain. He sketched the argument which von Neumann later used to show that the axiom of foundation was consistent relative to the other axioms.

7. Computing unsatisfiability

In 1920a Skolem proved the Skolem-Löwenheim theorem by invoking the axiom of choice. This is not too different from Löwenheim's own proof in 1915. Skolem's proof goes for formulas in Skolem normal form. If one assumes that the formulas are satisfiable in a domain D, then by using the axiom of choice we can find a countable subdomain E of D where the formulas are satisfiable. Löwenheim considered only a single formula, and the use of the axiom choice was formulated as a logical principle using some kind of index calculus. We could perhaps formulate it as the principle in higher order logic

$$\text{From } \forall x \exists y. Rxy \text{ infer } \exists f \forall x. Rxfx$$

But his idea was not that different from Skolem's (1920a). One problem here is that there is no constructive way of finding the subdomain E from D.

An important step was made in 1922b. There Skolem considered what is now called the term model. By starting with formulas in Skolem normal form one could give names to all individuals needed in the domain. He used the natural numbers as names here. In 1928a he started with Skolem functions for the $\exists$-quantifiers in the Skolem normal form. Then as names he had all terms built up from the Skolem functions starting with a term 0. Then he switched to using natural numbers instead. He used a simple enumeration of the Skolem terms to do that. The point is that Skolem saw the choice between natural numbers and the Skolem terms for names as just a matter of convenience.

So Skolem worked with what is now called the Herbrand universe—after Herbrand's later work. The main mathematical point in the computation of unsatisfiability is explicitly stated and proved in Skolem 1922b. He showed that if the formulas are unsatisfiable, then this is so already using a finite part of the Herbrand universe given by taking all Skolem terms of some finite nesting depth. He proved this using König's lemma as he should.

At the time this result was not recognized for its worth. Jacques Herbrand proved his result in 1929 and Kurt Gödel his in 1930. Skolem's 1922 result is better than Herbrand's. Herbrand's proof is not correct, but Skolem's is. Herbrand mentioned unification, but did not use it in his theorem. The next step for both Skolem and Herbrand would have been to use something like unification to cut down the number of terms that it was necessary to consider. Such an argument appeared with Dag Prawitz around 1960—and was much used by Alan Robinson in his resolution method from the mid-1960s. Gödel had the formulation of completeness as the equivalence of the syntactic concept "derivable" and the semantic concept "valid". This is a way of stating the problem about computing unsatisfiability which came up with the textbook of Hilbert and Ackermann from 1928. The mathematical content was cleared up with Skolem 1922a.

It is worth remarking that the usual proof of the completeness theorem in the textbooks of logic does not give as good a result as Skolem 1922a. Usually the completeness is proved by an argument given by Leon Henkin around 1950. Even although the argument is fairly elegant it does not give the same result as Skolem's argument does. This is perhaps most clearly seen if we look at the corresponding algorithms for unprovability. The Henkin argument does not give any more than the "British Museum Algorithm"—enumerate all possible proofs and search through them one by one until a proof is found that fits. The mathematical reason why this is so comes from the fact that Henkin tries to give totally defined models while Skolem gives only partially defined models—or, for the algebraically minded, Henkin gives maximal ideals while Skolem gives only prime ideals. So through his argument Henkin tries to put more properties in the potential models he considers—and therefore the algorithm connected with the nonexistence of models becomes less effective than it should. Gödel in his 1930 proof does much the same thing as Skolem did in 1922a. His underlying algorithm is equally good.

8. Computing satisfiability

Skolem was of course aware that his procedure did not usually terminate for formulas which were satisfiable. So the interesting question comes up in trying to say something about cases where something could be done for satisfiable formulas. An easy case is when the formulas are in $\exists\forall$–form. In 1928a Skolem showed that finite models would suffice here, and an upper estimate on the size of the model could be calculated from the formulas. The more interesting case is the $\forall\exists$–case, where there are no function symbols in the matrix of the formula and there is only one $\forall$-quantifier. Skolem showed in 1928a that one could tell from a finite part of the Herbrand universe whether the formulas were satisfiable or not. The Herbrand universe itself is usually infinite—so it requires an argument. The argument that Skolem gives is a combinatorial one. At the same time Wilhelm Ackermann proved the same thing with about the same argument.

Having started on showing how some satisfiability of some forms of formulas could be treated Skolem went on and proved in the 1930s that a number of other forms could also be treated.

9. Elimination of quantifiers

Consider Skolem's proof of his normal form theorem. Here he introduced new relation symbols. In some contexts this is not necessary. One could find quantifier-free formulas doing the same job. In 1919a he did this for the structure of subsets of a given set with subset-relation and operations like union, intersection and complement. This was the first time an "elimination of quantifier"-type argument was used. Much later it became one of the main methods of model theory. But rarely is Skolem mentioned as the pioneer.

The argument from 1919a is fairly straightforward. A more difficult argument comes in 1930c. He had proved that number theory with addition admitted elimination of quantifiers. But then Skolem heard that Alfred Tarski's doctoral student Presburger had proved the same thing. So Skolem went on in 1930c to prove that number theory with multiplication alone admitted elimination of quantifiers.

10. Primitive recursive arithmetic

In 1919 Skolem read Whitehead and Russell's *Principia mathematica.* He was annoyed that simple facts required long and cumbersome proofs. A simple fact like $1 + 1 = 2$ requires a few pages. Skolem took this as an indication that the foundation was not right. In the first decade after 1900 there was a debate about the foundation of mathematics between Poincaré and Hilbert. Hilbert wanted to found mathematics on some kind of axiom system. Later it was explained as an axiom system in first-order logic. Poincaré claimed that this could not be sufficient for number theory. It was necessary to require some understanding of the natural numbers. Otherwise we had a vicious circle. In Skolem 1923a he supports Poincaré's point of view. An understanding of the natural numbers was required, with wich one could give a system for developing number theory.

Skolem in fact gives two systems: One for defining objects, objects defined by primitive recursion, and one for proving properties of such objects. This would be quantifier-free statements with free variables, and with the possibility of proving theorems using (quantifier-free) induction. These two systems were developed in parallel. Skolem was able to develop a substantial part of number theory. For example, prime numbers were introduced.

Here Skolem comes very close to problems in contemporary computer science. Let us for the moment skip over the fact that Skolem considers only natural numbers. His two systems could be considered as a programming language—for defining objects—and a programming logic—for proving properties about the objects. This is in fact of great interest for contemporary computer science. Especially so if data structures other than natural numbers are admitted. Skolem's idea works equally well for data structures with more than one base element and more than one generator. —And so we are in business, with an interesting programming language and its corresponding programming logic. I believe that Skolem would have considered this as the same idea as in 1923a.

11. Elementary functions

Skolem's 1923a became known in the logic community through Hilbert and Bernays' famous book *Grundlagen der Mathematik* from 1934 and 1938. They used the primitive recursive functions in their formulation of the proof of Gödel's incompleteness theorem. A little later it was realized that not all primitive recursive functions were needed. In fact a better class of functions could be the elementary ones introduced by L. Kalmar in 1943. These are the functions constructed from constants, addition, multiplication, the Kronecker delta d and bounded sums and products. In fact, the whole development of Skolem 1923a could have been done with these functions.

Skolem came back to problems with the elementary recursive functions a number of times in the 1950s. It is not hard to construct primitive recursive functions going beyond the elementary ones. The function f defined by

$$f(0) = 1$$
$$f(n+1) = 2^{f(n)}$$

will do. But the problem remains whether such functions play a part in ordinary mathematics. Do we really need more than the primitive recursive functions? Perhaps even smaller classes of functions are equally interesting for practical purposes. Some of his last papers consider such problems. He defined the class of lower elementary functions in the same way as the Kalmar elementary functions except for not admitting bounded products. He showed that any recursively enumerable function could be achieved as an enumeration of such a simple function. The lower elementary functions are functions of polynomial growth—and I think that they should be studied more closely.

12. Non-standard models

From at least 1920 Skolem was sceptical of the use of predicate logic in characterizing mathematically useful structures. In 1922a he considered what he called set theoretic relativity. That is, the impossibilty of characterizing set theory in first-order logic. At that time he also seemed to have some thoughts about number theory. In 1928b he was more concrete. He considered the structure of polynomials in a variable x. We get a structure not too unlike the natural numbers themselves. We can for example order them by

$$f < g \text{ iff } \exists x \forall y > x.(fy < gy)$$

He was unable to get a structure elementary equivalent to number theory at once. He had to do some more work. In a number of papers we are able to see progress—until 1933d. There he proved that it was impossible to characterize number theory with axioms from first-order logic. For the result he had to assume an enumeration of all statements. Now the proof may be given using ultraproducts. The assumption about an enumeration would then correspond to the existence of an ultrafilter. Skolem's result follows of course from Gödel's incompleteness result. But Skolem's result and technique is of independent interest.

13. Conclusion

At the time of Skolem's death in 1963 logic was taking a direction away from Skolem's interest. This is clearly indicated in the discussions about his work in the *Selected Works in Logic*. But things have changed since then. Skolem, with his emphasis on the computational aspects of logic, is more important than ever.

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