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A BRIEF HISTORY OF THE NOTATION OF BOOLE'S ALGEBRA

1. Introduction

A typical class on programming in logic taught at the university touches on issues such as Boolean Algebra, De Morgan's Law, Herbrand bases, Peano Axioms, Hilbert-type calculus, Gödel's Undecidability Theorem, Turing Machines, Von Neumann Architecture, the Fixpoint Theorem of Knaster Tarski, Formulas in Skolem Normal Form, the Theorem of Löwenheim Skolem, Robinson's Unification Algorithm, and perhaps programming in Ada, Pascal, or Gödel. Though the contributors to nowadays' computational logic are remembered, the historical development itself tends to be forgotten in computer science. This is particularly sad, since the history of logic is long and vivid and can contribute much to the understanding of current problems.

The history of computational logic spans about 400 years and reaches from Leibniz's idea of a calculus for reasoning to today's logic-based programming languages. The idea of the mechanisation of human thought is not linear and it is interesting to view and compare the objectives and paradigms of its contributors: After Leibniz's project of a philosophical language, which tied for the first time philosophical reasoning to mathematics, it took nearly 200 years until England saw De Morgan's and Boole's work in logic. Subsequently, mathematical logic was established step by step and the discussion concerning the new foundation of mathematics at the beginning of the 20th century led to an outburst of work in the area.

In these developments of logic, our focus of interest is the creation of an algebra of logic by George Boole, because his work treated logic

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for the first time *formally*. Boole's ingenious ideas are revealed in particular in the good notation he employed. Usually little attention is paid to mathematical notation, but in general it is crucial in problem-solving to adopt the right notation that fully captures the essentials of the problem.

This article aims to survey the development of the notation underlying Boole's Algebra. It puts together bits and pieces to narrate a brief history of the notation of Boole's Algebra. Starting from the development of algebraic notation from Arabian times, it describes how the notation and laws were finally freed from their arithmetic interpretation. Furthermore, we sketch De Morgan's and Boole's lives and works and show how Boole connected algebra and logic.

2. Algebraic Notation

Muhammad ibn Mūsā al-Khwārizmī (780?–847?), who worked at Baghdad's "House of Wisdom", is often credited with being the father of algebra.[1] His book "Al-jabr wa'l muqābalah", which can be translated as "restauration and reduction", gives a straight-forward and elementary exposition of the solution of equations.

A typical problem, taken from chapter V, is the division of ten into two parts in such a way that "the sum of the products obtained by multiplying each part by itself is equal to fifty eight".[2] The solution, three and seven, is constructed geometrically in quite an elegant fashion. Besides his own methods, al-Khwārizmī uses procedures of Greek origin such as proposition 4 of book II in Euclid's Elements:

> If a straight line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangles contained by the segments.[3]

Take a look at Figure 1. Euclid's theorem states that the sum of the shaded squares and the two remaining rectangles is equal to the whole square. In the case of al-Khwārizmī's problem, the whole square has 100 units since the straight line has ten units. The two shaded squares on the segments have fifty-eight units and so al-Khwārizmī concludes that each rectangle amounts to twenty-one units. To complete the solution of the problem, we quote from Rosen's translation of al-Khwārizmī's Algebra:

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[1] Boyer and Merzbach 1992, p. 228.

[2] In today's notation the problem is to solve the equations $x+y=10$ and $x^2+y^2=58$.

[3] Katz 1993, p. 64. In today's notation the theorem states that $(x+y)^2 = x^2+2xy+y^2$.

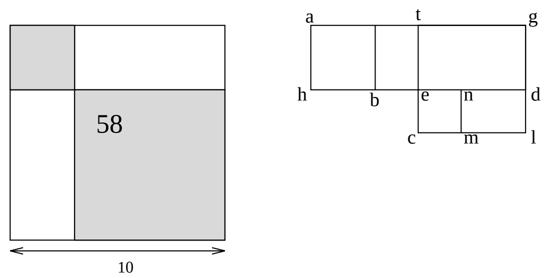

Figure 1: Geometric solution of equations.

What must be the amount of a square, which, when twenty-one dirhems are added to it, becomes equal to the equivalent of ten roots of that square? Solution: Halve the number of the roots; the moiety is five. Multiply this by itself; the product is twenty-five. Subtract from this the twenty-one which are connected with the square; the remainder is four. Extract its root; it is two. Subtract this from the moiety of the root, which is five; the remainder is three. This is the root of the square which you required, and the square is nine. Or you may add the root of the moiety of the roots; the sum is seven; this is the root of the square which you sought for, and the square itself is forty-nine.[4]

Because of the lack of any symbols and notation the problem and solution are difficult to understand. They are based on a figurative approach, as shown on the right in Figure 1. There, the base line is divided into two parts of equal length so that the rectangle *bt* equals *md*. Since *bg* covers twenty-one units we conclude that the gnomon *tenmlg* does too. Furthermore, we know that the square *cg* equals twenty-five units, so that *cn* equals four and *cm* two. Finally, we subtract two from five, the length of *ct*, and obtain the solution three and subsequently seven. Such a figurative solution is straight-forward but difficult to understand, since it does not use any notation for problem-solving.

The Arabian knowledge, being itself based on its Babylonian, Egyptian, Greek, and Indian roots, slowly spread through Italy over Europe. A milestone is the book "Liber Abbaci" by Leonardo of Pisa (1180?–1250), who is also known as Fibonacci. His book strongly influenced the French Nicolas Chuquet (1445–1488), who in 1484 composed a treatise entitled "Triparty en les science de nombres". Concerning notation, he wrote for example our $\sqrt{14 - \sqrt{180}}$ as $R^2\underline{14\bar{m}R^2 180}$ and $5x$, $6x^2$, and $10x^3$ as 5^1, 6^2, and 10^3, respectively. He even allowed zero and negative

[4]Rosen 1831, p. 11. In our notation the solution of $x^2 + 21 = 10x$ is asked for and $x = \frac{10}{2} \pm \sqrt{[(\frac{10}{2})^2 - 21]}$ is computed.

exponents so that $9x^0$ and $9x^{-2}$ was written as 9^0 and $9^{2\bar{m}}$. But the work was not printed until 1880, so it was of little influence.

At about the same time, in 1489, a German lecturer at Leipzig, Johann Widmann (1462–1498), published the commercial arithmetic "Rechnungen uff allen Kaufmanschafften", which first contained the symbols $+$ and $-$. These two symbols may have originated from marks chalked on chests of merchandise in German warehouses; they indicated the variation in the standard weight.[5] They were also used in the books "Die Coss", written in 1524 by the celebrated Rechenmeister Adam Riese (1492–1559), which promoted Hindu-Arabic numerals and calculation by pen instead of counting with an Abacus, and in Christoff Rudolff's (1499–1545) "Coss", published in 1525, which also first introduced the $\sqrt{}$-sign. The important equality sign was introduced in 1557 in the "Whetstone of Witte" by the Welsh Robert Recorde (1510–1558):

> And to avoide the tediouse repetition of these woordes :is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe [twin] lines of one lengthe, thus: $=\!=\!=$, bicause noe .2. thynges, can be moare equalle.[6]

Though algebraic notation advanced considerably, it was still only able to treat special cases. Here the French François Viète (1540–1603) introduced a convention as simple as it was fruitful: He used a vowel for unknown quantities and a consonant for known ones. But unfortunately, he did not use the advanced notation developed by his predecessors.

Beside $+$ and $-$, Simon Stevin (1548–1620) from Flanders used the division sign $\div$ in his book "Stelreghel", meaning algebra. The signs $<$ and $>$ in turn were introduced by the English Thomas Harriot (1560–1621) in his work "Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas", which was published in 1631. In the same year "Clavis Mathematicae" by William Oughtred (1574–1660) appeared, containing the multiplication sign $\times$.

For the first time all these notations and conventions were brought together by the French philosopher and mathematician René Descartes (1596–1650). His book "La geométrie", where he develops a geometric algebra, uses letters near the beginning of the alphabet for known values and letters at the end for unknown ones. He used the signs $+$ and $-$ and the exponential notation. As equality sign he wrote $\propto$. All in all, he is the first author whose writings are readable for us.

[5] Bowen 1995.

[6] Boyer and Merzbach 1992, p. 290.

3. Logic and Computation

After Descartes left the stage, Gottfried Wilhelm Leibniz (1646–1716), perhaps the last universal genius, entered it. Leibniz carried out tremendous work in philosophy, mathematics, law, politics, diplomacy, and history. One particular project, one that was also motivated by Leibniz's passion for world and religious peace,[7] concerned a philosophical language free of ambiguities. Thus it would help to facilitate human communication and understanding and would finally end all controversies that depended on reasoning.[8]

Such a philosophical language lies within the tradition of the search for a universal language, which was first generally described by Dante Alighieri in about 1303 in "De vulgari eloquentia".[9] The basic idea is that words coincide with the meaning they denote. And if the syntactical level reveals the semantics of the word, then the language must be understandable everywhere because its words are not arbitrary anymore.

However, Leibniz's project was even more ambitious. While the first three were never finished, Leibniz's great achievement was to combine the idea of a "lingua universalis" with a "calculus raciocinator". The two together facilitate "blind thinking", as Leibniz terms it, since reasoning is reduced to arithmetic calculation.

Leibniz assigned characteristic numbers to concepts. Basic concepts are assigned prime numbers and complex concepts non-primes, so that the composition of the complex concept out of basic ones is exactly revealed by the multiplication of primes giving a non-prime characteristic. Leibniz gives an example in his "Elementa Calculi" of 1679, where he assigns 2 to "animal" and 3 to "rational". He concludes that "human" is characterized by 2×3, i.e. 6, as humans are rational animals.[10] In a similar fashion, he argues that an ape having the characteristic 10 is not a human, since neither is 10 divisible by 6 nor 6 by 10. But both have 2, i.e. being an animal, in common.

Leibniz was well aware of the relation of his calculus to algebra and geometry; in "Elementa Calculi" he mentions explicitly that he uses letters to abbreviate and replace concrete objects as done in algebra and geometry.[11] Beside the operation of multiplication Leibniz denoted

[7] Eco 1994, p. 278.

[8] Eco 1994, p. 284.

[9] Eco 1994, p. 47.

[10] "Verbia gratia quia Homo est Animal Rationale hinc si sit Animalis numerus a ut 2, Rationalis vero numerus r ut 3, erit numerus Hominis seu h idem quod ar id est in hoc exemplo 2, 3 seu 6." (Herring 1992, p. 70)

[11] Herring 1992, p. 73.

an inclusive *or* by addition. In the paper "Non Inelegans Specimen Demonstrandi in Abstractis" ("A not inelegant example of abstract proving"), written about 1685–87, he even admits equations differing from arithmetics such as $A+A = A$, or $A+A\infty A$ as he put it.[12] Leibniz was far ahead of his time and still some 150 years later Boole, who first *formally* defined a logical algebra, did not admit such necessary laws.

Earlier, in 1677, Leibniz stated in a paper without title that he had invented an elegant trick to show the use of his calculus: He assumes "those marvellous characteristic numbers" as given.[13] To express it negatively, he was not successful at putting his calculus to work in practice, as he never tried to define the characteristic numbers. However, he was the first who tried to tie philosophy and mathematics together and who claimed that reasoning can be reduced to arithmetic calculation and thus be mechanised.[14]

Leibniz's work was only fully appreciated in the 20th century with Couturat's complete edition.[15] He was not read, particularly in England, because of an argument between Leibniz and Newton (1643–1727). Both had developed the infinitesimal calculus independently. Newton's method of fluxions, denoted by a dot, was motivated by physical considerations and his notation was quite unhandy. In contrast, Leibniz, who was more interested in defining an abstract and elegant calculus, developed the notation of differentials dx as used today. In about 1700 a controversy started, ending with Leibniz and Newton accusing each other of plagiarism. Thereupon British mathematics stuck to Newton's method and fell back behind Continental mathematics where Leibniz's notation led to fast progress.

4. British Mathematics in the First Half of the 19th Century

Some hundred years later the Edinburgh Review of 1808 summed up the situation as follows:

> In the list of mathematicians and philosophers to whom the science of Astronomy for the last sixty or seventy years has been indebted for its improvements, hardly a name from Great Britain falls to be mentioned (...) in the knowledge of higher geometry they were not on a footing with their brethren on the Continent. We will venture to say that the number of those in this island who can read the "Mécanique Céleste" with any tolerable facility is small indeed.[16]

[12] Herring 1992, p. 163.
[13] Herring 1992, p. 57.
[14] Davis 1983.
[15] Herring 1992, p. VIII.
[16] MacHale 1985, p. 45.

Until about 1830 Paris was the only mathematics center and education in mathematics was generally in a poor state throughout Europe.[17] But the first half of the 19th century marks a political and mathematical turning point. From about 1760 the industrial revolution began to change British society dramatically. England was called the "workshop of the world" having in 1850 the world's greatest coal output. The new inventions, such as the steam engine and the railway, and the growing enterprises produced a great need for engineering skills and indirectly also for the education of mathematics. In the second half of the 19th century Great Britain, then a wealthy and relatively safe empire, reacted with the formation of many colleges and universities.[18]

In general, European mathematics was turning from an outsiders' science to a professional one: In the 1830s, journals such as "Journal Mathématique Pures et Appliqué", "Journal für die reine und angewandte Mathematik" and the "Cambridge Mathematical Journal" were initiated and new mathematics centers in Berlin, Göttingen, and Cambridge developed. Cambridge played a key role in the change of paradigm that overcame Britain's neglected symbolic access to mathematics. At Trinity College, a school with a long tradition, having been formed in 1546 by Henry VIII, the algebraist George Peacock (1791–1858), the astronomer John Herschel (1792–1871) and Charles Babbage (1791/2–1871) founded the "Analytical Society" in 1813. It was initiated by Babbage as a parody to bible meetings that were being held by that time in Cambridge, and the members wanted to leave "the world wiser than they found it". But besides students' jokes the society seriously aimed at promoting "the principles of pure *d*-ism [Leibniz] as opposed to the *Dot*-age [Newton] of the University"[19] and their translation of Sylvestre Lacroix's (1765–1843) "Traité elementaire de calcul différential et de calcul integral" was important and strongly influenced British mathematics, which was about to change fundamentally.

The new interest in symbolic manipulation opened new avenues in symbolic algebra. An important step towards a separation of algebra and arithmetic was taken by Peacock, who in his 1830 "Treatise on Algebra", argued:

> Whatever form is algebraically equivalent to another when expressed in general symbols, must continue to be equivalent whatever these symbols denote.

and conversely:

[17] Meschkowski 1990, p. 11f.
[18] Frerichs, Bode, Gerbstedt, and Killian 1988, pp. 22f.
[19] Babbage 1990, p. 21.

Whatever equivalent form is discoverable in arithmetical algebra considered as the science of suggestion when the symbols are general in their value, will continue to be an equivalent form when the symbols are general in their nature as well as in their form.[20]

But he failed to abstract a general principle of symbol manipulation, as he still believed "that no views of the nature of Symbolical Algebra can be correct or philosophical which made the selection of its rules of combination arbitrary and independent of arithmetic".[21]

Peacock's treatise influenced Augustus De Morgan (1806–1871), who also studied at Trinity College and who went a step further than Peacock. De Morgan believed that one could create an algebraic system with arbitrary symbols and a set of laws under which these symbols were manipulated. Only afterwards would one give an interpretation of the laws and manipulations. In 1849 he explained this principle in his "Trigonometry and Double Algebra":

Given symbols M, N, $+$, and one sole relation of combination, namely that $M + N$ is the same as $N + M$. Here is a symbolic calculus: how can it be made a significant one? In the following ways, among others. 1. M and N may be magnitudes, and $+$ the sign of addition of the second to the first. 2. M and N may be numbers, and $+$ the sign of multiplying the first by the second. 3. M and N may be lines, and $+$ the direction to make a rectangle with the antecedent for a base, and the consequent for an altitude. 4. M and N may be men and $+$ the assertion that the antecedent is the brother of the consequent. 5. M and N may be nations, and $+$ the sign of the consequent having fought a battle with the antecedent.[22]

George Boole (1815–1864), who developed his ideas independently of the new spirit of Cambridge, had already as a 19-year-old in 1835 quite similar ideas:

As to the lawfulness of this mode of procedure, it may be remarked as a general principle of language, and not of the peculiar language of Mathematics alone, that we are permitted to employ symbols to represent whatever we choose that they should represent—things, operations, relations, etc., provided 1st, that we adhere to the signification once fixed, 2nd, that we employ the symbols in subjection to the laws of the things for which they stand.[23]

Before we present Boole's logical framework that resulted from these ideas, we give a brief overview of De Morgan's and Boole's lives.

[20] Peacock 1834, p. 199.
[21] Katz 1993, p. 614.
[22] De Morgan 1849b, p. 94.
[23] MacHale 1985, p. 50.

5. De Morgan's Life and Work

Augustus de Morgan was born as fifth child on the 27th of June 1806 in Madura, India, where his father worked as an officer for the East India Company. His family soon moved to England, where they lived first at Worcester and then at Taunton. His early education was in private schools, where he enjoyed a classic education in Latin, Greek, Hebrew, and mathematics. In 1823, at the age of 16, he entered Trinity College in Cambridge, where the work of the "Analytical Society" had already changed the students' schedule so that De Morgan also studied Continental mathematics.

In 1826, he graduated as a fourth Wrangler and turned his back on mathematics to study to be a lawyer at Lincoln's Inn in London. But only a year later he revised this decision and applied for a position as professor of mathematics at the newly established University College in London. At the age of 22, with no publications, he was appointed.

The work that De Morgan produced in the years to come spanned a wide variety of subjects with an emphasis on algebra and logic. But surprisingly he was not able to connect them. An important work of his was the "Elements of Arithmetic", published in 1830, containing a simple yet thorough philosophical treatment of the ideas of number and magnitude. In a paper from 1838 he formally described the concept of mathematical induction and in 1849 in "Trigonometry and Double Algebra" he gave a geometrical interpretation of complex numbers.

De Morgan's first work in logic is dated 1839 and is entitled "First Notions in Logic". In contrast to Boole, De Morgan was a traditional logician who knew the medieval theory of logic and semantics. Today, he is mainly known for De Morgan's Law, which states that the negation of a conjunction/disjunction is equal to the disjunction/conjunction of the negated conjuncts/disjuncts (and which was already known by William of Ockham in the 14th century[24]). But his main contributions are found in the theory of syllogisms, where he was the first since medieval times extensively to discuss quantified relations. He recognized that relational inferences were the core of mathematical inference and scientific reasoning.

His logical papers are published in a series entitled "On the Syllogism", and the first four can also be found in his book "Formal Logic; or, the Calculus of Inference, Necessary and Probable", published in 1847. In these works, he uses letters X, Y, Z, etc. to stand for arbitrary general terms or names "which it is lawful to apply to any one of a collection of objects of thought: and in the language of Aristotle,

[24] EB 1995, De Morgan.

A	All Xs are Ys	$X)Y$
O	Some Xs are not Ys	$X:Y$
E	No Xs are Ys	X,Y
I	Some Xs are Ys	XY

Figure 2: De Morgan's notation for the categorical forms $AOEI$.

that name may be predicated of each of these objects".[25] An important novelty in his work is the introduction of a "universe of a proposition, or of a name" that "may be limited in any manner expressed or understood" in contrast to a fixed universe of all things used since Aristotle. De Morgan does not use an explicit operator for negation, but instead denotes the "contrary" of a name X by x. The conjunction of proposition P and Q is expressed as PQ and the disjunction by P,Q. This notation captures De Morgan's Law as "the contrary of PQ is p,q" and "the contrary of P,Q is pq". De Morgan seemed unaware of the importance of a good notation, because he chose nearly arbitrarily the symbols to represent the traditional categorical form $AOEI$, as shown in Figure 2, and the symbols for E and I coincide even with disjunction and conjunction. It is obvious that this inappropriate notation limited De Morgan's work.

Besides his work in algebra and logic, De Morgan contributed 712 articles to the "Penny Cyclopedia". He worked on the history of mathematics and wrote biographies on Newton and Halley and a dictionary of mathematicians of the 17th century. He was also engaged in many associations. In 1828, he became a member of the "Royal Astronomy" and 3 years later he helped to found the "British Association for the Advancement of Science". He ran the "Society for the Diffusion of Useful Knowledge", and when he retired from his academic post in 1866 he was involved in the foundation of the "London Mathematical Society" and became its first president. His diversity also covered educational subjects: He wrote essays on mathematical education, the concept of an École Polytechnique and the education of the deaf and dumb.

De Morgan's professional career was not smooth and he proved to have a strong and sometimes eccentric personality in academic life. For example, in 1831, three years after he was appointed professor, he resigned because a colleague was dismissed without explanation. Five years later, he regained the chair, when his replacement died in an accident. As a student, De Morgan never applied for a fellowship at

[25] De Morgan 1849a.

Oxford or Cambridge, since he refused to submit the necessary religious tests. Later, he never had his name put forward to the "Royal Society" and rejected an Honours Degree from the University of Edinburgh. In his whole life he never voted in elections.

6. Boole's Life

This section is based on MacHale's biography (1985). George Boole was born in Lincoln on November 2nd, 1815, as the first child of John Boole, a shoemaker, and Mary Ann Joyce, a lady's maid. His talents were visible already in his young years. Besides primary education, Boole was taught by his father in English and elementary Mathematics. Together they built cameras, kaleidoscopes, microscopes, telescopes, and a sundial. At the age of ten Boole's Latin was so good that a tutor, a friendly bookseller, was hired. Besides Latin, Boole mastered Greek completely on his own. The Booles' financial situation did not allow for a secondary education. But they managed to pay for a period at a commercial academy, where Boole spent his spare time reading English literature and learning French, German and, later, Italian.

At the age of 16, Boole had to start to work to support his parents and his three sisters since his father's business collapsed. He was employed as an assistant teacher in a school at Doncaster, some 40 km away from Lincoln. During this period Boole devoted much of his time to the reading of French mathematics, such as Joseph-Louis Lagrange's (1736–1813) "Mécanique Analytique", Pierre-Simon Laplace's (1749–1827) "Mécanique Céleste" and later Lagrange's "Calcul de Fonction", Karl Jacobi (1804–1851), Siméon Poisson (1781–1840) and last but not least Newton's "Principia". But with the intense self-education, he neglected his profession and he was regarded as a bad teacher by his colleagues.

After Doncaster, Boole taught for half a year in Liverpool, which was farther away and where he felt even lonelier than in Doncaster. Finally, as his parents' health gave cause for concern and his salary was still the only one to support the family, he returned to Lincoln and opened—at barely 19 years of age—his own school. Boole ran his school quite successfully and was able to support his parents and sisters. While his educational skills developed through experiments and experience in his school, he continued his studies, spending many reading hours in Lincoln's Mechanics' Institute, a forerunner of adult education and trade unions. Both Boole and his father were members of the institute, whose objective was "the cultivation of Experimental, Natural and Moral Philosophy; and of useful knowledge in all departments—

avoiding Politics and controversial Divinity." Besides consulting the institute's library, Boole received many books from the friendly Sir Edward ffrench Bromhead, Fellow of the Royal Society, who lived not far from Lincoln. Based on his reading, Boole started his own investigations and two years later he published his first article, "Researches on the Theory of Analytical Transformations, with a special application to the Reduction of the General Equation of the Second Order", in the second issue of the "Cambridge Mathematical Journal". In the same year, three more articles followed, tackling a variety of themes, such as "On Certain Theorems in the Calculus of Variations", "On the Integration of Linear Differential Equations with Constant Coefficients" and "Analytical Geometry".

With these publications Boole attracted the attention of the leading British mathematicians. It has to be kept in mind that Boole was an outsider in the mathematical community and not acquainted with his famous contemporaries. But this was about to change. In 1842, Boole introduced himself to De Morgan, who was already a well-established mathematician. After this, both exchanged their thoughts in many letters and developed a lifelong friendship.

A milestone in Boole's career was his paper "On a General Method in Analysis", which was published in the "Philosophical Transactions of the Royal Society". Boole's insecurity when he submitted the paper as a newcomer was justified when the Royal Society almost rejected the paper without consideration, since he was not among the acknowledged contributors of British mathematics and socially not on the level of the Society's members. But, fortunately, due to the insight of the Edinburgh mathematics professor Kelland, the paper finally received its deserved merits: it was printed and Boole was awarded the Gold Medal of the Royal Society.

Already by 1846 Boole had 14 publications, most of which appeared in the "Cambridge Mathematical Journal" and, encouraged by De Morgan, he applied for a position as professor at the Queen's Colleges, which were to be founded then. The application ran for a long time, and feeling without hope, Boole even withdrew it for a short time. But, finally, in 1849, he was appointed professor of mathematics at the Queen's College of Cork, Ireland.

Boole's environment in Cork stood in great contrast to his personal life. Between 1845 and 1850, Ireland was haunted by the great famine that decreased the population by one third. Furthermore, it was the religious war fought in Ireland and at the college that made Boole unhappy. But personally, Boole got along quite well. Being a bachelor until his mid-thirties, he finally married the much younger

Mary Everest (1832–1916) and had five daughters by her.

While Mary's mathematical knowledge did not progress substantially after her marriage, Boole was about to collect the merits of his work. Between 1850 and 1853 he had worked much in probability theory and in 1854 he wrote his major book "An Investigation into the Laws of Thought", a mature and settled work based on his earlier book on logic and exposing the connection between logic and probability. Also in 1854 he was elected president of the Curvian Society and three years later he was accepted as Fellow of the Royal Society and won the Keith Prize. In 1859 and 1860 Boole published two more books, "Differential Equations" and "Finite Differences".

7. Boole's Work

During Boole's time in Doncaster as assistant teacher he had a flash one afternoon as he was walking across a field, and the initial idea of a logic of human thought was born. Fourteen years later, early in the spring of 1847, his interest in logic was re-awakened with a quarrel raging between De Morgan and Sir William Hamilton, the Scottish Philosopher and Metaphysician, concerning the origin of the quantification of the predicate. Hamilton extended classical logic, which dealt with propositions of the form "all A are B, no A are B, some A are B, some A are not B", to a quantification of the second term, as in "all A are all B, any A is not some B". A similar approach was taken by De Morgan and consequently Hamilton accused De Morgan of plagiarism, though the idea in question was not new.

The controversy attracted much attention and, in particular, it inspired Boole's work on logic, as he states in the preface of his 1847 book:

> In presenting this Work to public notice, I deem it not irrelevant to observe that speculations similar to those which it records have, at different periods, occupied my thoughts. In the spring of the present year, my attention was directed to the question then moved between Sir W. Hamilton and Professor De Morgan; and I was induced by the interest which it inspired, to resume the almost-forgotten thread of former inquiries. It appears to me that, although Logic might be viewed with reference to the idea of quantity, it had also another and deeper system of relations. If it was lawful to regard it from without, as connecting itself through the medium of Number with the intuitions of Space and Time, it was lawful to regard it from within, as based upon facts of another order which have their abode in the constitution of the Mind.[26]

[26] Boole 1847, Preface.

He wrote the book, which was published in 1847, on the same day as De Morgan's "Formal Logic", at a furious pace, and so it contained several flaws which led him—seven years later—to write the corrected, more settled book "An Investigation into the Laws of Thought". In the sequel we review the logic he developed in the "Laws of Thought".

Concepts are independent of their representation: "Romans expressed by the word 'civitas' what we designate by the word 'state'. But both they and we might equally well have employed any other word to represent the same conception".[27] Besides such *literal symbols* which stand for concepts, he introduces operations to relate and manipulate the concepts. The only relation Boole uses is equality. The similarity to Leibniz's idea of the "calculus ratiocinator" is striking, but Boole did not know Leibniz's work.

Proposition I.

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz:

1st. Literal symbols, as x, y, &c., representing things as subject of our conceptions.

2nd. Signs of operations, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements.

3rd. The sign of identity, $=$.

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.[28]

Boole's idea of a mind selecting items from classes is central in his arguments for the laws that hold in his algebra. For example, to get all "good men" the mind selects from the class "men" those who possess the further quality "good". Obviously, the order of the selection does not affect the result, so that Boole argues that $xy = yx$, the commutative law, holds.[29] In a similar way, he proves that the distributive law holds. Interestingly, he also uses the associative law, but never mentions it explicitly. As arithmetic provides the laws $0y = 0$ and $1y = y$ he states that 0 is nothing and 1 is the universe. In his first book, Boole still assumes a fixed universe of all things in existence.

[27] Boole 1854, p. 26.
[28] Boole 1854, p. 27.
[29] Boole 1854, p. 29.

$x+y=y+x$	$xy=yx$	Commutative Law
$(x+y)+z=x+(y+z)$	$(xy)z=x(yz)$	Associative Law
$x+0=x$	$x1=x$	Neutral Element
$x+(-x)=0$		(Additive) Inverse
	$x(y+z)=xy+xz$	Distributive Law
	$x^2=x$	
	$0\neq 1$	
$x+x=0$ implies $x=0$		

Figure 3: Boole's Algebra

Influenced by De Morgan's universe of discourse, he adopts this idea in the "Laws of Thought". Though the universe may possibly be empty, Boole excludes this trivial case[30] and implicitly assumes that the law $0 \neq 1$ holds. The idea of selecting from classes also leads him to a law that holds in arithmetic only for 0 and 1, namely $x^2 = x$. Together with an additive inverse and the rule that $x + x = 0$ implies $x = 0$, Boole's Algebra, as summarized in Figure 3, is completely specified including also the laws he used but which he did not mention explicitly.

As an illustration Boole uses the definition of wealth due to the economist N.W. Senoir: "Wealth consists of things transferable, limited in supply, and either productive of pleasure or preventative of pain." With w representing wealth, t transferable things, s limited in supply, p productive of pleasure and r preventative of pain, Boole obtains the equation:

$$w = st(p + r(1 - p)) \qquad \text{or} \qquad w = st(p(1 - r) + r(1 - p))$$

depending on whether the "or" in the definition is read inclusively or exclusively.

In order to cope with statements such as "all men are mortal", or "all men are *some* mortal beings", Boole introduces a special symbol v,

> a class indefinite in every respect but this, viz., that some of its members are mortal beings, and let x stand for 'mortal beings', then will vx represent 'some mortal beings'. Hence if y represents men, the equation sought will be $y = vx$.

[30]Hailperin 1986, p. 84, in case $0 = 1$ the laws lose their intuitive meaning since all classes are empty.

> ... it is obvious that v is a symbol of the same kind as x, y, etc., and that it is subject to the general law $v^2 = v$, or $v(1-v) = 0$.[31]

Later we will see the problems and misunderstandings that the symbol causes.

Boole solved equations guided by the three principles of *division*, *development*, and *interpretation*. Probably he transferred the idea from differential equations, where he enhanced a solution method presented by Duncan Farquharson Gregory (1813–1844) in 1840.[32] As an example for the solution of equations, he presents the "definition of 'clean beasts' as laid down in the Jewish law, viz., 'clean beasts are those which both divide the hoof and chew the cud' "[33], i.e.

$$x = yz,$$

where x represents the clean beasts, y beasts dividing the hoof and z beasts chewing the cud.

In order to obtain the relation in which 'beasts chewing the cud' stand to 'clean beasts' and 'beasts dividing the hoof' we divide by y:

$$z = \frac{x}{y}$$

which is developed by a case analysis of x and y. Each summand is one of the four cases for bindings of x and y and consists of the fraction with the values replaced and multiplied by the case:

$$\begin{aligned} z &= \tfrac{1}{1}xy + \tfrac{1}{0}x(1-y) + \tfrac{0}{1}(1-x)y + \tfrac{0}{0}(1-x)(1-y) \\ &= xy + \tfrac{1}{0}x(1-y) + 0(1-x)y + \tfrac{0}{0}(1-x)(1-y) \end{aligned}$$

The equation can be read as

> Beasts which chew the cud [z] consists of all clean beasts (which also divide the hoof) [xy] together with an indefinite remainder (some, none, or all) [indicated by $\frac{0}{0}$] of unclean beasts which do not divide the hoof [$(1-x)(1-y)$].[34]

Thus, the terms $0(1-x)y$ and $\frac{1}{0}x(1-y)$ are omitted. The former because it does not make a statement at all, the latter because it does not make a statement about z. By multiplication with 0, the equation

[31] Boole 1854, p. 61.
[32] Hailperin 1986, p. 66.
[33] Boole 1854, p. 86.
[34] Boole 1854, p. 87.

$z = \frac{1}{0}x(1-y)$ can be rewritten as $0z = x(1-y)$, i.e. there are no clean beasts that do not divide the hoof.

No doubt, Boole's notation was ingenious and the work was developed in a formal style, which was a very important innovation, but the work also produced some unintuitive or even wrong results. In the next section we review De Morgan's criticism of Boole's work.

8. De Morgan's Criticism of Boole's Work

In 1847 Boole and De Morgan both wrote books on logic, which were published on the very same day in November of 1847. While De Morgan was more concerned with the traditional syllogistic view of logic, Boole presented a very clear and elegant algebraic approach to logic. After the quarrel with Hamilton, De Morgan asked Boole not to exchange ideas concerning logic, since he wanted to prevent another accusation of plagiarism. But on the 28th of November, when both books were published, De Morgan compared their work in a letter sent to Boole:

I am much obliged to you for your tract, which I have read with great admiration. I have told my publisher to send you a copy of my logic which was published on Wednesday.

There are some remarkable similarities between us. Not that I have used the connexion of algebraical laws with those of thought, but that I have employed mechanical modes of making transitions, with a notation which represents our head work.

For instance, to the notation of my Cambridge paper I add

XY name of everything which is *both* X and Y
X, Y name of everything which is either X or Y.

Take your instance of p. 75

$$x = y(1-z) + z(1-y).$$

I express your data thus

$$1 ... X)Zy, Yz \qquad Zy, Yz)X ... 2$$

The following is all rule, helped by such perception as beginners have of the rules which will succeed in solving an equation

From 1. $\qquad$ $notX = x$, etc.

$$\{z, Y\}\{y, Z\})x$$
$$zy, zZ, Yy, YZ)x$$

But zZ, yY are nonexistent

$$
\begin{array}{lll}
 & zy, YZ)x & \\
\text{or} & YZ)x \quad \text{or} & YZ)xY. \\
 & \text{But from 2} & yZ)X \\
 & \quad \text{or} & yZ)Xy \\
 & \quad YZ, yZ)Xy, xY & \\
 & \quad Z)Xy, xY &
\end{array}
$$

But by 1.

$$
\begin{array}{lll}
Xy)Zy, Yzy & \text{or} & Xy)Zy \\
XY)ZyY, Yz & \text{or} & XY) \quad \text{[De Morgan deleted this line]} \\
 & \text{by 2} & x)\{z, Y\}\{y, Z\} \\
 & & x)zy, ZY \\
 & & xY)xyY, ZY \\
 & \text{or} & xY)ZY \\
 & & Xy, xY)Zy, ZY \\
 & \text{or} & Xy, xY)Z \\
 & \text{or} & Xy, xY \text{ and } Z \text{ are identical.}
\end{array}
$$

This is far from having the elegance of yours; but your system is adapted to identities, in mine an identity is two propositions. Perhaps I should pass from

$$X)Zy, zY$$

to

$$Z)X, xY$$

more readily than you would. But I am not sure.

In fact there hang a multitude of points upon this question whether complex or simple forms are to come first.[35]

While De Morgan tackles their notation and representation in the letter, he also prepared a draft on the very same day, which he did not send. In the draft he points out the difficulties of division and as an example he states that the equation $zx = zy$ does not imply $z = 0$ or $x = y$ as true in arithmetics. From today's point of view the problem is quite simple. Though Boole assumes an additive inverse, he does not mention a multiplicative inverse, so that the division is not well-defined. The draft carefully examines this problem:

The solution of the elective equations will, I have no doubt, be found inexpugnable. With regard to the syllogistic process, there are unexplained difficulties about v and about division by y. Here you have recourse to verbal monitions about the meaning of v. The process of division is not *per se* allowable.

[35] Smith 1982, p. 24.

$xz = yz$ does not give $x = y$. Take page 35

$$\begin{array}{rcll} y & = & vx & \\ 0 & = & zy & \\ y \times 0 & = & vxzy & \text{admitted} \\ 0 & = & vxzy & \text{do.} \end{array}$$

Now you may separate

$VZ.XY$ in my notation
No VZ is XY
But not No VZ is X
and yet $VX.ZY$ give $VX.Z$

There is something to explain about the division by y.

I think with Mr Graves that $y = vx$ is the primitive form. But v is not a definite elective symbol, make it what you know it to be, and I think the difficulty vanishes

$$\begin{array}{rcl} y & = & yx \\ 0 & = & zy \\ y \times 0 & = & zxy^2 \\ 0 & = & zxy \end{array}$$

Now some Zs are not Xs, the ZYs. But they are *nonexistent*. You may say that *nonexistents* are not Xs. A nonexistent horse is not even a horse; and, (*à fortiori?*) not a cow. This is not suggested by your paper; but appears in my system.

I see that 0 must be treated as a magnitude in form $y \times 0/y$ is 0: but $0/y$ is not capable of interpretation.

In fact, your inverse symbol is not interpretable, except where use of the direct symbol has preceded.

xy	make a mark on all the Ys which are Xs
$\frac{1}{x}(xy)$	Rub them out again
$\frac{1}{x}(y)$	Rub out marks which never were made -

But I do not despair of seeing you give meaning to this new kind of negative quantity.

It may be thus

$$0 = zxy$$

on the other side as

$$(xy)z = 0$$

is an equation of condition giving in my notation $XY.Z$ or $XY)z$ or Xz or $X : Z$. But in the form $(yz)x$ it is an identical equation, since $yz = 0$.

In $(zx)y$ it is true also though no conclusion to a syllogism, since the middle term is not eliminated.

Observe that the conclusion of the syllogism really is

Those Xs which are Ys are not Zs

Quaere, is there not even another process of reasoning before we arrive at the ordinary conclusion namely Those Xs which are Ys are not necessarily all Xs

Xs (not necessarily all) are not Zs

Or, is not syllogistic reasoning twofold in inference, on form and on quantity.[36]

The use of the symbol v was widely criticised, and Boole himself solved equations by first getting rid of the vs. For example, he used $y(1-x) = 0$ for $y = vx$. But if we want to recover the one from the other we get the equation $y = \frac{0}{1-x}$, which is developed to $y = 0(1-x) + \frac{0}{0}x$. But the fraction $\frac{0}{0}$ is read as some, none or all rather than v's some.

A fundamental law in Boole's Algebra is $x^2 = x$. Boole accepts the law $x^n = x$ in the "Mathematical Analysis of Logic", but in the "Laws of Thought" he corrects it and rejects the law. For the equation $x^3 = x$ he argues that it can be factorized such that terms as $1 + x$ and -1 occur. But they are not interpretable. For the rejection of the first term, he gives the intuitive argument that "we cannot conceive of the addition of any class x to the universe", and for the latter, the formal argument that "-1 is not subject to the law $x(1-x) = 0$, to which all class symbols are subject".[37] He did not realize that similar arguments apply to the law $x^2 = x$, which can be written as $x(x-1) = 0$.

A further drawback concerns addition, which is not closed, as he rejects both $1+1 = 0$, because it yields $(-1)^2 = -1$, and also $1+1 = 1$, because it implies an empty universe, i.e., $1 = 0$. From a nowadays point of view Boole could have solved all these problems by defining a Boolean ring with unit,[38] in which $1 = -1$ and $x + x = 0 \rightarrow x = 0$ hold. However, such a rigorous formal treatment was a second step taken after Boole's achievements.

Conclusions

In this article we set out on a long journey from the Arabian beginnings of algebraic notation to its use in logic in 19th-century Britain. The development of algebra undertook two important steps. First, the notation of algebra was developed; the centuries it took clearly shows how difficult a task it was. In a second phase the algebraic notation and laws were separated from their interpretation in arithmetics and

[36] Smith 1982, p. 26.
[37] Boole 1854, p. 50.
[38] Hailperin 1986, p. 84.

applied to other domains. The statements of Peacock, De Morgan, and Boole reveal the struggle of this abstraction.

Though De Morgan was strong in both algebra and logic, he was unable to connect them. His logical work was limited by his traditional background in logic and the inappropriate notation he used. In contrast, Boole tied logic and mathematics together, which led Bertrand Russell to say that "pure mathematics was discovered by Boole in a work called 'The Laws of Thought' ". During the preparation of this book, Boole expressed this vision in an inaugural address when he was dean of the faculty:

> I speak here not of the mathematics of number and quantity alone, but a mathematics in its larger, and I believe, truer sense, as universal reasoning expressed in symbolical forms, and conducted by laws, which have their ultimate abode in the human mind. That such a science exists is simply a fact, and while it has one development in the particular science of number and quantity, it has another in a perfect logic.[39]

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[39] MacHale 1985, p. 99.

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