MULTIPLEX SEMANTICS FOR DEONTIC LOGIC

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This multiplex semantics incorporates multiple relations of deontic accessibility or multiple preference rankings on alternative worlds to represent distinct normative standards. This provides a convenient framework for deontic logic that allows conflicts of obligation, due either to conflicts between normative standards or to incoherence within a single standard. With the multiplex structures, two general senses of "ought" may be distinguished, an indefinite sense under which something is obligatory when it is enjoined by some normative standard and a core sense for when something is enjoined by all normative standards. Multiple normative standards may themselves be given a preferential order; this leads to a concept of ranked obligation. This paper presents the foundations of this multiplex semantics and the propositional deontic logics they define.

1. Introduction: The Logic P

The present research grows primarily out of concern for the possibility of genuine conflicts of obligation, situations in which statements OA and $O\neg A$ might both be true. Secondarily, I am interested in the pattern of preference-based semantics for deontic logic, which, as we shall see, can accommodate conflicts of obligation when the more familiar Kripke-style relational semantics cannot.

To admit the possibility of conflicts of obligation requires significant changes in the structure of deontic logic from what is commonly assumed. The principle

D) $$OA \rightarrow \neg O \neg A,$$

which explicitly precludes conflicts of obligation, must, of course, be rejected. But this is not enough. Any logic which contains all of

a) $\vdash (A \wedge \neg A) \rightarrow B$
b) If $\vdash A \rightarrow B$, then $\vdash OA \rightarrow OB$
c) $\vdash (OA \wedge OB) \rightarrow O(A \wedge B)$

will necessarily contain

d) $\vdash (OA \wedge O\neg A) \rightarrow OB$

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which says that if there is any conflict of obligation, then everything is obligatory. That is going too far. To find oneself in a moral or normative dilemma is not suddenly to find that one must do everything. I am interested in the possibility of genuine conflicts that do not generate the catastrophic explosion of the entire normative structure.

To admit that possibility thus requires rejecting not only (D), but also either (a), (b) or (c). To reject (a) is to turn to a paraconsistent logic. I explored one way to do this in my paper (1999), which examined standard deontic principles coupled with the relevant logic R. Here I want to investigate the result of rejecting (c) while keeping the whole of classical logic for the underlying propositional logic.[1]

To reject (c) requires a non-normal deontic logic. It requires giving up, or significantly modifying, the familiar Kripke-style relational semantics for the deontic language, based on models $M = \langle W, R, v \rangle$ (in which W is a non-empty set of points, R is a binary relation on W, and $v(p) \subseteq W$), with the evaluation rule, for each $a \in W$:

O-Rule *R*) $M, a \models_R OA$ iff $M, b \models_R A$, for all b such that Rab.

If R is required to be serial, we have, of course, the standard semantics for standard deontic logic, SDL, including postulate (D), but even without seriality, all of (a), (b) and (c) are validated, and so we are still left without an adequate account of deontic dilemmas.

There is, however, another familiar way of interpreting deontic statements OA that does accommodate giving up (c) as well as (D). This is what may be called "preference-based semantics".[2] The leading idea is that alternative worlds may be compared with respect to value and ranked as being normatively better or worse, more or less ideal, without presuming

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[1] There is a large literature on the question of deontic conflicts, which cannot be surveyed in a paper like this. Here I am primarily concerned to present results for one approach to a deontic logic that allows for the possibility of such conflicts. It is not my intent to argue for that possibility, nor for the advantages of this approach over others, e.g., turning to a nonmonotonic logic.

[2] Pioneering work on this sort of semantics for deontic logic was done by Sven Danielson (1968), Bas van Fraassen (1972), and David Lewis (1973, esp. §5.1, also 1974), though this method was developed primarily for dyadic deontic logic. Similar preference relations also appear in the work of Bengt Hansson (1969), but with a different rule for evaluating deontic statements. More recently, forms of preference-based semantics have been used by quite a number of deontic logicians, although often as part of efforts to modify standard deontic logic significantly, and thus to define the modelling relation differently. Cf., for example, my own 1989, 1990a, 1990b, or Sven Hansson's 1990, 1997, or Leon van der Torre's 1997. Semantics for dyadic deontic logic can be applied to monadic statements via the postulated equivalence between OA and $O(A/\top)$.

that any are in any respect completely ideal or the normatively best of all possible worlds. More formally, our preference-based models are structures $M = \langle W, P, v \rangle$, with W and v as usual, and P assigns to each $a \in W$ a binary relation $\geqslant_a$ on W where "$b \geqslant_a c$" says that b ranks at least as good and perhaps better than c, from the point of view of a. Formulas OA are then evaluated according to the rule

O-Rule P) $M, a \models_P OA$ iff $M, b \models_P A$, for some $b \in \mathscr{F} \geqslant_a$, such that $M, c \not\models_P A$ for any $c \geqslant_a \mathscr{F} b$.

(where $\mathscr{F} \geqslant_a$ is the field of $\geqslant_a$, i.e., $\{b : \exists c(b \geqslant_a c \text{ or } c \geqslant_a b)\}$). Thus, we say that A is obligatory just in case there is a point at which it holds such that there is no situation in which it doesn't hold that is of equal or greater value.

In all that follows, we require that $\mathscr{F} \geqslant_a$ be non-empty for every $a \in W$. Relations $\geqslant_a$ that are (i) reflexive, (ii) transitive, and (iii) (strongly) connected on their fields will be called *standard*, and models all of whose assigned relations $\geqslant_a$ are standard will likewise be called standard. It is well known that standard deontic logic SDL is sound and complete with respect to the class of all such standard models.

It is condition (iii), the connectedness of standard ranking relations, that validates the consistency postulate (D) and excludes the possibility of conflicts of obligation. Hence, to admit this possibility we must allow for models whose relations do not meet this condition. Then it is easy to see how conflicts could occur. Given, for example, $W = \{a, b, c\}$ with just $b \geqslant_a b$ and $c \geqslant_a c$ and neither $b \geqslant_a c$ nor $c \geqslant_a b$ (and neither $a \geqslant_a b$ nor $a \geqslant_a c$), A might hold at b while $\neg A$ holds at c, and then OA and $O\neg A$ would both hold at a. In such models not only is (D) not valid, our target (c) above is also not valid. (Nor is (d), although (a) and (b) are.) Thus, we see the opportunity for a deontic logic that allows for conflicts of obligation while not reducing normative structures to triviality.

The general class of such preference-based models, including those whose assigned relations are not standard, determines the logic I call P. This may be formalized by these axioms and rules:

PC) If A is a classical tautology, then A is an axiom of P

RM) If $\vdash A \rightarrow B$, then $\vdash OA \rightarrow OB$

N) $\vdash O\top$

P) $\vdash \neg O \neg \top$

where $\top$ is any tautology. The whole is closed under *modus ponens* as well as (RM). We note that the general rule of necessitation (RN): if $\vdash A$ then

$\vdash OA$, is derivable, as are the rules: If $\vdash A \rightarrow B$, then $\vdash PA \rightarrow PB$, and if $\vdash A$, then $\vdash PA$, where PA abbreviates $\neg O \neg A$. Notice that P plus (C) = $(OA \wedge OB) \rightarrow O(A \wedge B)$ is exactly SDL.[3]

It is a small matter to prove that P is sound with respect to the class of all P-models. Completeness is considerably more difficult. In what follows I will sketch the proof since I refer to this result below, and also because it prompts the introduction of the multiplex semantics that is the chief topic of this paper. As far as I know, this is a new result. (Although P may easily be shown complete with respect to a neighborhood semantics, that result does not automatically carry over for the semantics based on preference rankings on worlds, and so it is necessary to reestablish completeness in this framework in order to be certain that P fully captures the present mode of understanding deontic statments.) Likewise, the later completeness theorems below are also all new results. Some do not even have counterparts in ordinary neighborhood semantics.

2. Multiplex Models

As noted, the logic P does not have a simple Kripke-style relational semantics. Nevertheless, it can be adequately characterized by a natural generalization of that, in what I call the multi-relational semantics. This is the first form of the multiplex semantics of the title.

A *multi-relational model* is a structure $M = \langle W, \mathscr{R}, v \rangle$ where W and v are as before and $\mathscr{R}$ is a non-empty set of serial binary relations R on W. Formulas OA are now evaluated according to the rule:

O-Rule MR) $M, a \vDash_{MR} OA$ iff there is a relation $R \in \mathscr{R}$ such that $M, b \vDash_{MR} A$, for all b that Rab.

(For a convenient notation, I will henceforth write

$$M, Ra \vDash_{MR} A$$

to abbreviate

$$\text{For all } b \text{ if } Rab \text{ then } M, b \vDash_{MR} A$$

[3] P is closely akin to the system van Fraassen (1973) introduced under the name D, similarly to accommodate conflicts of obligation. Van Fraassen gave essentially a neighborhood semantics for his system. The chief difference between P and D, aside from the semantics, is that P contains (N) and D does not. This does not seem an important difference to me. While some, reasonably, object to (N) and the corresponding rule (RN), van Fraassen's D avoids these only because it accepts models in which nothing, including $\top$, is obligatory. There are no models which allow some (contingent) things to be obligatory while denying that $\top$ is, which is what critics of (N) and (RN) would want.

so that this rule says OA holds at a iff there is an R in $\mathscr{R}$ such that $M, Ra \vDash_{MR} A$.)

THEOREM 1. *P is sound and complete with respect to the class of all serial multi-relational models.*

As usual soundness is routine and may be left to the reader. I sketch a proof of completeness in the Appendix.[4]

Just as we may think of the single relation R of the standard Kripke semantics for SDL as picking out those possible worlds, b, that are best or ideal with respect to a world a, so now in the multi-relational semantics, we may regard each relation in the set $\mathscr{R}$ as representing a particular standard of value and picking out those worlds, b, that are best with respect to a from the perspective of that standard, while recognizing that there could be other standards according to which b is not ideal. One common way to account for conflicts of obligation, to accommodate which is the primary motivation for the system P, is to suppose that when both OA and $O\neg A$ are true it is because A is prescribed by one set of norms or regulations while $\neg A$ is prescribed by another, distinct set. In the present picture, this is to say that A holds in all the worlds that are best according one relation in $\mathscr{R}$, while $\neg A$ holds in all the worlds that are best according to another relation in $\mathscr{R}$. Of course, within this picture, there will be no conflicts of obligation under a single standard or relation. Each set of norms or regulations is presumed to be internally consistent, and conflicts only emerge as a result of rivalry between sets of norms.

To accommodate the possibility that a particular system of norms might not be internally consistent, in the sense that its own prescriptions might generate conflicts of obligation, we must again move beyond the pattern of relations of deontic accessibility to something like the preference-based semantics described above. This is the second form of our multiplex semantics, a multi-preference-based semantics, in which there are multiple relations that rank worlds and do not presume that any are entirely acceptable or best.

Let a multi-preference-based model M be a structure $\langle W, \mathscr{P}, v\rangle$ with W and v as usual, and $\mathscr{P}$ assigning each $a \in W$ a non-empty set $\mathscr{P}_a$ of non-empty binary relations $\geqslant_a$. Deontic formulas, OA, are now interpreted

[4] Notice that here, and in all the results below, although for convenience I speak of soundness and completeness with respect to a class of *models*, I could as well say soundness or completeness with respect to the corresponding class of *frames* since no limitations are put on the assignment functions v.

according to the rule:

O-Rule MP) $M, a \models_{MP} OA$ iff there is a relation $\geqslant_a \in \mathscr{P}_a$ such that $M, \geqslant_a a \models_{MP} A$

where the notation

$$M, \geqslant_a a \models_{MP} A$$

abbreviates

there is a $b \in \mathscr{F} \geqslant_a$ such that $M, b \models_{MP} A$ and $M, c \models_{MP} A$ for any $c \geqslant_a b$

corresponding, for a given relation $\geqslant_a$, to the original evaluation condition in the (mono) preference-based semantics.

So far we put no restrictions on the multiple relations $\geqslant_a$ assigned to a except that their fields not be empty. We could require that they all be standard, which would correspond closely with the MR-semantics described above and would model the situation in which every set of norms was internally consistent. Or we could allow that some, or all, not be connected on their fields (or, for that matter, that they also not be reflexive or transitive), which would model situations in which a set of norms could be internally inconsistent, in the sense of generating conflicts of obligation by itself (without at the same time degenerating to triviality). For some purposes below we will want to limit our attention to the class of standard MP-models; for other purposes we will want to consider the general class of MP-models, including those that are not standard in this sense. The logic P itself is characterized by both classes, as well as the intermediate cases where all the assigned ranking relations are reflexive or transitive, but perhaps not connected, on their fields.

Theorem 2. *P is sound and complete with respect to (a) the class of all MP-models, (b) the class of all reflexive or transitive MP-models, and (c) the class of all standard MP-models.*

Again soundness is routine to establish, and the completeness proof is indicated in the Appendix.

While Theorems 1 and 2 show two multiplex ways of characterizing the logic P, they do not establish what we originally wanted, that P is the logic characterized by the simple (mono) preference-based semantics, without restrictions on the properties of the preference-ranking. Nevertheless, we can convert any multi-preference-based model into an equivalent non-multiple preference-model—see the Appendix for a sketch of how to do this—and that will establish

Theorem 3. *P is sound and complete with respect to the class of all (mono) P-models.*

(for if *A* is not provable in P then by Theorem 2 there is an MP-model on which it does not hold, hence *A* does not hold on that model's conversion, so *A* is not P-valid). Furthermore, the proof of Theorem 3 reveals that

COROLLARY 1. *P is complete with respect to the class of all* (*mono*) *P- models that are reflexive or transitive.*

3. GENERALIZED "OUGHT"

Although the multi-relational and multi-preference-based models defined in the preceding section were introduced as a technical device to establish the completeness of P with respect to the original (mono) preference-based semantics, their interest extends much further.

The multiple relations of the multiplex models may be thought to represent different normative standards; each defines a specific sense of "ought". The language of our deontic logic could contain distinct deontic operators to express each of these senses, but that is not necessary, and we shall not pursue such a multi-modal logic here. Nonetheless, one might naturally think of the "ought" defined through the multiplex rules of Section 2, as ambiguous between these many specific senses determined by each normative standard. *OA* says that it ought to be the case that *A*, but it does not specify under which sense of "ought". It says only that it ought to be the case that *A* under *some* system of norms. Appeals to ambiguity are often plausible ways to account for apparent inconsistencies, even deontic conflicts. The multiplex semantics is made for that kind of account.[5]

Nevertheless, there is another way of thinking of ambiguity than indefiniteness among many senses. That is to take a sentence "it ought to be that *A*" that does not specify "ought" with respect to what, to express not merely that it ought to be that *A* under *some* normative standard or other, but that it ought to be that *A* under *any* standard. Undifferentiated statements *OA* would thus be taken to express the core that is common to all the specific senses of "ought". It would say that it ought to be that *A* under *all* sets of norms or regulations.

This brings us to quite a different rule of interpretation for formulas *OA* within the multiplex semantics (an interpretation that is only distinctive given the multiplicity of the multiplex semantics). In order to distinguish this sense of "ought" from the preceding, let us now mark the deontic

[5] Such a view is presented in Schotch and Jennings 1981, pp. 156ff., with a multi-relational semantics similar to that developed here. This leads to the same system P for *O*. (Cf. also Jennings and Schotch 1981.)

operators. That operator O previously described and evaluated according to the MR and MP rules of Section 2, will henceforth be written "O_e", with "e" to indicate the existential quantification of its evaluation rule. The new operator will be written "O_a" for its evaluation rule will employ universal quantification. That is, to catch the sense that "A is obligatory" is true just in case A is enjoined by every set of norms or regulations, we stipulate for the multi-relational semantics:

O_a-Rule MR) $M, a \vDash_{MR} O_a A$ iff $M, Ra \vDash_{MR} A$ for every $R \in \mathscr{R}$

and similarly, for the multi-preference-based semantics:

O_a-Rule MP) $M, a \vDash_{MP} O_a A$ iff $M, \geqslant_a a \vDash_{MP} A$ for all $\geqslant_a \in \mathscr{P}_a$

I will call O_a the "core" sense of "ought" in contrast to the "indefinite" sense that is O_e.

It is notable that if all relations in the set $\mathscr{R}$ of MR models are serial, or all relations in the sets $\mathscr{P}_a$ of MP models are standard, then O_a behaves according to the principles of SDL even while O_e follows P.

Let us make this more precise. For a language L_{ae} that contains just the two monadic modal operators O_e and O_a (in addition to what is required for classical propositional logic), let L_{ae} be interpreted by means of multi-relational or multi-preference-based models in which formulas $O_e A$ are evaluated according to the MR or MP rules of Section 2 and formulas $O_a A$ are evaluated according to the rules just stated. The logic for the pair of modalities O_a and O_e that is thus characterized by the class of all serial MR-models or all standard MP-models, is defined by PC plus

K_a)	$O_a(A \to B) \to (O_a A \to O_a B)$
D_a)	$O_a A \to \neg O_a \neg A$
RN_a)	If $\vdash A$ then $\vdash O_a A$
RM_e)	If $\vdash A \to B$ then $\vdash O_e A \to O_e B$
N_e)	$\vdash O_e \top$
P_e)	$\vdash \neg O_e \neg \top$
K_{ae})	$O_a(A \to B) \to (O_e A \to O_e B)$

The first three postulates plainly give us SDL for O_a while the second three give us P for O_e. Hence I call this logic $SDL_a P_e$. The last postulate links the two operators.

THEOREM 4. *$SDL_a P_e$ is sound and complete with respect to (a) the class of all serial MR-models, and (b) the class of all standard MP-models.*

Proof of this is contained in the proof of Theorem 6 described in the Appendix.

The condition that MR-models be serial and the MP-models be standard is significant. That O_a satisfies the principles of SDL requires the presumption that each of the normative standards reflected in the multiple relations of the multiplex be individually coherent and not admit conflicts of obligation by itself. That presumption seems as artificial in the present context as when made in setting of a single relation as in Section 1. To allow individual normative standards to be internally incoherent means, however, once again moving away from the pattern of the relational Kripke semantics, even within the multiplex framework. The reason is the same as before; to encounter a normative dilemma with respect to a normative standard does not necessarily entail the complete destruction of that standard. In the framework of the multi-preference-based semantics we can, though, relax the requirement that every ranking relation in the sets $\mathscr{P}_a$ be standard, and in particular we can allow that they not be connected on their fields. This will have no effect on the basic logic of the indefinite sense of obligation, O_e; that remains P. For the core sense of obligation, O_a, as one would predict, its logic now also follows P rather than SDL. Thus, the principles (K_a) and (D_a) are no longer valid without connectivity. In addition, the mixed postulate (K_{ae}) is also no longer valid. Instead, the combined logic P_aP_e will have the postulates (N_a) $O_a\top$ (P_a) $\neg O_a \neg\top$, and (RM_a). If $\vdash A \to B$ then $\vdash O_aA \to O_aB$, reflecting now the P-character of O_a, as well as (N_e), (P_e), and (RM_e), for the P-character of O_e, and, in place of (K_{ae}), it will have

O_aO_e) $\qquad\qquad\qquad\qquad O_aA \to O_eA$

to connect the two.

THEOREM 5. *P_aP_e is sound and complete with respect to the class of all MP-models (or reflexive or transitive MP-models).*
(See the proof of Theorem 7 sketched in the Appendix.)

It is noteworthy that the logic of the operator O_a thus distinguishes between the understanding of conflicts of obligation that is based on incompatibilities between distinct, but internally coherent normative standards, and the understanding of conflicts that allows that they could arise also from individual standards being inconsistent. O_e cannot do this, since its logic is P in either case.

It is also worth noting that this system P_aP_e has a semantics only in terms of multi-preference-based models. Unlike its component P_e, it does not have multi-relational semantics (under the core interpretation for statements O_aA). Indeed, it does not even have an ordinary neighborhood semantics. This is evident from the fact that any multi-relational model, or any neighborhood model, will automatically validate $(O_aA \wedge O_aB) \rightarrow O_a(A \wedge B)$, which is not provable in $\mathrm{P}_a\mathrm{P}_e$, without any conditions on the models that might now be relaxed. Since $\mathrm{P}_a\mathrm{P}_e$ seems a very natural system, combining, as it does, the core sense of obligation with the indefinite sense when neither is immune to conflicts of obligation, this shows an important application of the framework of multiple preference-based relations that lies beyond the reach of the more familiar frameworks of multiple accessibility relations or even classical neighborhoods.

4. Ranked Relations

Given multiple sets of normative standards, such as different systems of laws or systems of rules of organizations, or combinations of these, it is natural to think that there might be an order of priority on them. For example, one set of laws might be superior to another, as federal law is, perhaps, to local law, or authorities in an organization may stand in a hierarchical order so that the rules emanating from a higher office carry more weight than the rules from a lower one, as the President's instructions might outweigh the Dean's within a university (and the law might trump them both).

If we take the multiple accessibility relations within the MR-semantics or the multiple preference rankings of MP-models to represent the structure of evaluation of such distinct sets of norms, then we might suppose these relations themselves to be ordered to reflect such precedence. Accordingly, let us extend the former MR-models, $M = \langle W, \mathscr{R}, v\rangle$, by also assigning to each $a \in W$ a binary relation $\leqslant_a$ on $\mathscr{R}$, a ranking among accessibility relations. That is, let a MR$\leqslant$-model, M, be a structure $\langle W, \mathscr{R}, \leqslant, v\rangle$ where W, $\mathscr{R}$, and v are just as before and $\leqslant$ is a function that assigns to every $a \in W$ a binary relation $\leqslant_a$ on $\mathscr{R}$ that is reflexive and transitive. (We do not require connectivity for this relation since not all normative standards are comparable.) Similarly, in the multi-preference-based semantics, extend MP-models to form MP$\leqslant$-models, $M = \langle W, \mathscr{P}, \leqslant, v\rangle$, where W, $\mathscr{P}$, and v are as before and $\leqslant$ is a function that assigns to every $a \in W$ a binary relation $\leqslant_a$ on $\mathscr{P}_a$ that is reflexive and transitive. Initially, let us suppose that every relation in each $\mathscr{P}_a$ is standard; later we may drop this requirement.

Relations $\leqslant_a$ define an order of precedence among normative standards; this may be expressed in the language of a deontic logic with the addition of a new binary connective, $\leq$, such that $A \leq B$ is well-formed when A and B are. Such statements are evaluated so that they are true just when for any normative standard under which A is obligatory, there is a standard under which B is obligatory that is at least as, and perhaps more, important.

More formally, let $L_{ae\leq}$ be the language that results from adding this new connective to the language L_{ae} above. Then, given a MR$\leqslant$-model $M=\langle W, \mathscr{R}, \leqslant, v\rangle$ let

$\leq$-Rule MR$\leqslant$) $M, a\models_{MR\leqslant} A\leq B$ iff for every relation $R\in\mathscr{R}$ such that $M, Ra\models_{MR\leqslant} A$, there is an $S\in\mathscr{R}$ for which $M, Sa\models_{MR\leqslant} B$ and $R\leqslant_a S$

and similarly, given an MP$\leqslant$-model $M=\langle W, \mathscr{P}, \leqslant, v\rangle$, let

$\leq$-Rule MP$\leqslant$) $M, a\models_{MP\leqslant} A\leq B$ iff for every $\geqslant_a^i\in\mathscr{P}_a$ such that $M, \geqslant_a^i a\models_{MP\leqslant} A$, there is a relation $\geqslant_a^j\in\mathscr{P}_a$ for which $M, \geqslant_a^j a\models_{MR\leqslant} B$ and $\geqslant_a^i\leqslant_a\geqslant_a^j$

both of which reflect the connective $\leq$'s representing a ranking of normative standards. Within this language statements O_eA and O_aA are evaluated exactly as before.

The logic that is determined by these models extends SDL$_a$P$_e$ with the following principles containing $\leq$:

$O_a\leq$) $O_a(A\rightarrow B)\rightarrow(A\leq B)$

$\neg O_e\leq$) $\neg O_eA\rightarrow(A\leq B)$

$\leq O_e$) $(A\leq B)\rightarrow(O_eA\rightarrow O_eB)$

trans) $(A\leq B\wedge B\leq C)\rightarrow(A\leq C)$

and so we shall call this logic SDL$_a$P$_e\leq$.[6]

Given the other postulates of SDL$_a$P$_e\leq$, more expected principles are derivable, such as

reflex) $A\leq A$

R$\leq$) If $\vdash A\rightarrow B$ then $\vdash A\leq B$

both of which come from (RN$_a$) and ($O_a\leq$).

THEOREM 6. *SDL$_a$P$_e\leq$ is sound and complete with respect to* (*a*) *the class of all serial MR$\leqslant$-models*; *and* (*b*) *the class of all standard MP$\leqslant$-models.*
(The proof is sketched in the Appendix.)

Earlier, at the end of Section 3, we allowed the multiple preference relations in an MP-model to lack connectivity, and thus moved from the

[6] Although we have come to it in quite a different way, this system is equivalent to Mark Brown's 1966 system *CO* of comparative obligation augmented by (N$_e$), an option Brown considers. The present modelling for deontic formulas is quite different, however, as Brown applies a neighborhood semantics for formulas corresponding to our O_aA and O_eA, and a sort of hyper-neighborhood semantics for formulas $A\leq B$.

system SDL_aP_e to the system P_aP_e. So now we consider the system $P_aP_e\leq$ that results from the same relaxation of conditions on the preference relations in MP$\leqslant$-models.

Axiomatically, this will be the system that results by adding to P_aP_e the postulates $(\neg O_e\leq)$, $(\leq O_e)$, (trans) and $(R\leq)$ together with

$O_a\leq)'$ $\qquad\qquad O_aA\rightarrow B\leq A$

in place of $(O_a\leq)$ of $SDL_aP_e\leq$, which is no longer valid ($(R\leq)$ must also be postulated separately since it is no longer derivable from the other postulates).

Theorem 7. *$P_aP_e\leq$ is sound and complete with respect to the class of all MP$\leqslant$-models (or the class of all reflexive or transitive MP$\leqslant$-models).*
(Again, see the Appendix for the proof.)

Like P_aP_e, this extension $P_aP_e\leq$ has only a multi-preference semantics; it does not have an equivalent multi-relational or neighborhood semantics.

While the formal model structures for formulas $A\leq B$ are clear, there remains a question of how best to read such formulas in ordinary language. They do not say that B is at least as good as A; $\leq$ is not a connective of preference *per se*. That would be better understood with respect to the individual preference relations of P- or MP-models, and, in MP-models, it would admit both an indefinite and a core sense just as "ought" does. It is tempting to read $A\leq B$ in terms of comparative obligation, as saying that B is at least as obligatory as A. But this is not quite right either. US federal law requires me not to tamper with a US mailbox, my state's law requires me not to murder my neighbor, but we would not say that it is more, or even at least as, obligatory that I not tamper with a mailbox than that I not murder my neighbor even though federal law has priority over state law.[7] The best we seem able to say is that $A\leq B$ says that B is obligatory under a normative system that is at least as important as any that A is obligatory under. This may be useful with regard to some conflicts of obligation, not to resolve the conflict, but to give guidance in practical reasoning.[8]

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[7] This was pointed out by Paul McNamara and others in discussion at DEON'00. The reading of $A\leq B$ in terms of comparative obligation seems more appropriate under Brown's 1996 semantics for such formulas in terms of neighborhoods and sets of neighborhoods, where each neighborhood is taken to represent an obligation itself, and not simply a normative structure that gives rise to obligations.

[8] It may also be useful to think of merging multiple normative structures, and the priority of one over another may play a significant role in that process, especially in cases of conflicting regulations. Laurence Cholvy and Frederic Cuppens develop such a system in their 1999.

5. Summary and Further Research

The results presented here are very basic. They establish the completeness of the logic P with respect to the simple preference-based semantics for deontic logic and with respect to the two forms of multiplex semantics. With the multiplex semantics we have the resources to distinguish two general senses of "ought", the indefinite sense and the core sense, and also to introduce the notion of ranked obligation. Completeness is established for logics that represent these concepts, thus demonstrating the adequacy of the multiplex semantics itself.

The underlying purpose and theme throughout this discussion has been to develop deontic logic that allows for conflicts of obligation or normative dilemmas, and to do so in a way that follows conventional methods for interpreting deontic statements. Thus, the multi-relational semantics generalizes from the standard Kripke-semantics for deontic logic, and the multi-preference semantics generalizes from the familiar pattern of preference-based interpretions. The multi-preference semantics is of particular importance since it applies to logics that relational semantics cannot reach, notably P_aP_e and $P_aP_e\leq$, that allow not only that one normative standard might be inconsistent with another, but also that there might be normative standards that are internally inconsistent, and generate conflicts of obligation out of themselves. After all, this seems a genuine possibility.

These results are only established for the simplest sort of monadic propositional deontic logic, despite the well known limitations of such simple systems. My purpose, however, has been foundational, to lay the groundwork for further developments that could go in a number of directions.

Originally, the pattern of preference-based semantics was introduced to provide interpretations not for SDL at all, but for *dyadic* deontic logics, especially logics of conditional obligation. (See the references in Note 2.) It would be very natural to extend the present multiplex frameworks to apply to such a logic. In particular, it would be worthwhile investigating the effects of not requiring connectivity of the preference ranking(s). Presumably a weak dyadic logic would result that would correspond to P much as standard dyadic deontic logics stand to SDL, but precisely what that would be remains to be seen. Within the framework of multiplex semantics, one could also distinguish an indefinite sense and a core sense of conditional obligation. Perhaps one could extend that to incorporate a sense of ranked conditional obligation. It is, however, the nature of the dyadic logics to be significantly more complicated than their monadic counterparts, and we may expect that to hold all the more so with these extensions.

Much of the criticism of SDL has pointed to the severe limits of its expressive power, and much of the work of contemporary deontic logic has been to enrich the base of its language. While dyadic deontic logic is one direction that has taken, we also see, for example, attention paid to temporal aspects of norms, to the agents who are enjoined to act and to the authorities that might institute norms, and to the very idea of action and agency. The combination of deontic and alethic modalities may also be important. For the most part deontic logic is investigated at the propositional level, yet most recognize the need for full first-order theories, which might, or might not, raise the same sort of questions that arise in other sorts of quantified modal logics.

Very often, when such extensions are proposed, they assume the underlying pattern of SDL. Nevertheless, even with the additional complications of these extensions, the question of conflicts of obligation remains, and so perhaps ultimately the extended logics should be grounded on something like P rather than SDL. Within these extended frameworks the contrast between the indefinite and the core senses of obligation may also deserve attention. The multiplex semantics should prove a fruitful foundation for such investigation.

Appendix

Here I sketch proofs of the completeness theorems claimed in the main text. Unfortunately, because of limits on length, it is not possible to present all the details. Instead I point out the most salient steps so that the reader might reconstruct the full proofs for him- or herself. (Complete details should appear elsewhere.)

THEOREM 1. *P is complete with respect to the class of all serial multi-relational models.*

Proof: This follows familiar Henkin-style methods. Define a canonical model for P, $M = \langle W, \mathscr{R}, v \rangle$, where W is the class of all maximal consistent extensions of P and $v(p) = \{a : a \in W \text{ and } p \in a\}$, and $\mathscr{R} = \{R : \exists A(R = R^A)\}$ when, for every formula A, $R^A = \{\langle a, b \rangle : \text{either } OA \notin a \text{ or } A \in b\}$. Obviously $\mathscr{R}$ is non-empty, and it is easy to show that every $R \in \mathscr{R}$ is serial given the presence of (P) in P. Hence M is a serial multi-relational model. Next one shows that, for every A and every $a \in W$, $A \in a$ iff $M, a \vDash_{MR} A$. This is by induction on A, and we may take it for granted for every A up to $A = OB$. (a) In case $OB \in a$, it is virtually trivial to show that $M, a \vDash_{MR} OB$ given $R^B \in \mathscr{R}$ and the inductive hypothesis. (b) In case $M, a \vDash_{MR} OB$, then there is an $R = R^C$ in $\mathscr{R}$, for some C, such that $M, R^C a \vDash_{MR} B$, i.e., for all b such that $R^C ab$, $M, b \vDash_{MR} B$. It follows that $\vdash C \rightarrow B$, else $\{C, \neg B\}$ would be consistent and there would be a maximal consistent extension of $\{C, \neg B\}$, call it c. By consistency $B \notin c$, so by the inductive hypothesis B does not hold at c. Hence not-$R^C ac$; hence it is not the case that either $OC \notin a$ or $C \in c$; i.e., $OC \in a$ and $C \notin c$. But it is given that $C \in c$, a contradiction. Since $\vdash C \rightarrow B$, $\vdash OC \rightarrow OB$ by (RM). Either $OC \in a$ or $OC \notin a$. In the first case, $OB \in a$, by *modus ponens*. In the second case, $\vdash B$, else $\{\neg B\}$ is consistent and has a maximal consistent extension c, and automatically $R^C ac$, so $M, c \vDash_{MR} B$ and $B \in c$, by the inductive hypothesis, contrary to the consistency of c. Since $\vdash B$, $\vdash OB$ by (RN), hence $OB \in a$, as required. Given that $A \in a$ iff $M, a \vDash_{MR} A$ for all A, then the theorem follows as usual.

THEOREM 2. *P is sound and complete with respect to* (*a*) *the class of all MP-models*, (*b*) *the class of all reflexive or transitive MP-models*, *and* (*c*) *the class of all standard MP-models*.

Proof: This follows from Theorem 1. Given any MR-model, $M = \langle W, \mathscr{R}, v \rangle$, let its derived MP counterpart be $M^* = \langle W, \mathscr{P}, v \rangle$ where W and v are the same, and $\mathscr{P}$ assigns to each point a the set of relations $\mathscr{P}_a$ defined as follows: for each $R \in \mathscr{R}$, let $\geqslant_a^R = \{\langle b, c \rangle : Rab \text{ or not-}Rac\}$, and let $\mathscr{P}_a = \{\geqslant_a : \exists R(R \in \mathscr{R} \text{ and } \geqslant_a = \geqslant_a^R\}$. This set will be non-empty and each relation in it will be reflexive, transitive and connected. M^* is an MP-model. Furthermore, it is easy to demonstrate, and so may be left to the reader, that for every A and every $a \in W$, $M, a \vDash_{MR} A$ iff $M^*, a \vDash_{MP} A$. This proves Theorem 2 since if a formula A were valid on every MP-model but not provable in P, by Theorem 1, there would be an MR-model on which A does not hold, and so A would not hold on that model's derived MP counterpart, a contradiction. Since the relations of the derived MP counterpart are reflexive, transitive, and connected on their fields and their fields are all of W, this applies to all three parts of Theorem 2.

THEOREM 3. *P is sound and complete with respect to the class of all* (*mono*) *P-models*.

Proof: This follows from Theorem 2, but somewhat circuitously. Given an arbitrary MP-model $M = \langle W, \mathscr{P}, v \rangle$ in which every $\geqslant_a \in \mathscr{P}_a$ is reflexive on its field, let each relation $\geqslant_a \in \mathscr{P}_a$ bear a distinct index, i, and let I_a be the set of those indices. Let I be the union of all the sets I_a for $a \in W$. For convenience, $\geqslant_a^i$ is the relation in $\mathscr{P}_a$ that bears the index i. From M we define a (mono) P-model $M^* = \langle W^*, P^*, v^* \rangle$ as follows: Let W^* be the set of all pairs $\langle a, i \rangle$ such that $a \in W$ and $i \in I$. For each relation $\geqslant_a^j \in \mathscr{P}_a$, define for each $a \in W$ and each $i \in I$ a relation on W^*, $\geqslant^{*j}_{\langle a,i \rangle}$, such that

$$\geqslant^{*j}_{\langle a,i \rangle} = \{\langle \langle b, k \rangle, \langle c, l \rangle \rangle : k = l = j \text{ and } b \geqslant_a^j c\}$$

and let $\mathscr{P}^*_{\langle a,i \rangle}$ be the set of such $\geqslant^{*j}_{\langle a,i \rangle}$ for all $j \in I_a$. Let

$$\geqslant^*_{\langle a,i \rangle} = \bigcup \mathscr{P}^*_{\langle a,i \rangle}$$

and let P^* assign $\geqslant^*_{\langle a,i\rangle}$ to $\langle a, i\rangle$. Finally let

$$v^*(p) = \{\langle a, i\rangle : a \in v(p) \text{ and } i \in I\}$$

It should be obvious that M^* is a P-model. Moreover,

Observation. *For all* $\langle a, i\rangle \in W^*$, (1) $\geqslant^*_{\langle a,i\rangle}$ *is reflexive on its field, and* (2) *if every relation* $\geqslant^j_a \in \mathscr{P}_a$ *is transitive, then* $\geqslant^*_{\langle a,i\rangle}$ *is transitive.*
(These follow from the corresponding properties for M.)

Lemma 1. *For all* $a \in W$ *and all* $i \in I$, $M, a \models_{MP} A$ *iff* $M^*, \langle a, i\rangle \models_P A$.

Proof: By induction on A. Obvious when $A = p$. We show only the case when $A = OB$. (i) Suppose $M, a \models_{MP} OB$, i.e., there is a $\geqslant^j_a \in \mathscr{P}_a$ such that there is a $b \in \mathscr{F}\geqslant^j_a$ and $M, b \models_{MP} B$ and for all c that $c \geqslant^j_a b$, $M, c \models_{MP} B$. By reflexivity, $b \geqslant^j_a b$, so $\langle b,j\rangle \geqslant^{*j}_{\langle a,i\rangle} \langle b,j\rangle$ and $\langle b,j\rangle \geqslant^*_{\langle a,i\rangle} \langle b,j\rangle$, and hence $\langle b,j\rangle \in \mathscr{F}\geqslant^*_{\langle a,i\rangle}$. Moreover, $M^*, \langle b,j\rangle \models_P B$, by the inductive hypothesis. Suppose then some $c^* \in W^*$ that $c^* \geqslant^*_{\langle a,i\rangle} \langle b,j\rangle$. There is then a $\geqslant^{*m}_{\langle a,i\rangle} \in \mathscr{P}^*_{\langle a,i\rangle}$ such that $c^* \geqslant^{*m}_{\langle a,i\rangle} \langle b,j\rangle$. $c^* = \langle c, k\rangle$ for some $k \in I$. Since $\langle c, k\rangle \geqslant^{*m}_{\langle a,i\rangle} \langle b,j\rangle$, $k = j = m$ and $c \geqslant^m_a b$. So $c \geqslant^j_a b$, whence $M, c \models_{MP} B$, and $M^*, \langle c, k\rangle \models_P B$, by the inductive hypothesis, which suffices for $M^*, \langle a, i\rangle \models_P OB$. (ii) Suppose $M^*, \langle a, i\rangle \models_P OB$, so that there is a $b^* \in \mathscr{F}\geqslant^*_{\langle a,i\rangle}$ such that $M^*, b^* \models_P B$ and for all $c^* \in W^*$, if $c^* \geqslant^*_{\langle a,i\rangle} b^*$ then $M^*, c^* \models_P B$ too. With $b^* \in \mathscr{F}\geqslant^*_{\langle a,i\rangle}$, there is a relation $\geqslant^{*j}_{\langle a,i\rangle} \in \mathscr{P}^*_{\langle a,i\rangle}$ such that $b^* \in \mathscr{F}\geqslant^{*j}_{\langle a,i\rangle}$, which implies that $b^* = \langle b,j\rangle$ for some $b \in W$, and $\langle b,j\rangle \geqslant^{*j}_{\langle a,i\rangle} \langle b,j\rangle$ by the Observation above, and so $b \geqslant^j_a b$ and $b \in \mathscr{F}\geqslant^j_a$. Further, $M, b \models_{MP} B$, by the inductive hypothesis. Suppose then a c that $c \geqslant^j_a b$. By definition, $\langle c,j\rangle \geqslant^{*j}_{\langle a,i\rangle} \langle b,j\rangle$, and so $M^*, \langle c,j\rangle \models_P B$, whence $M, c \models_{MP} B$, again by the inductive hypothesis. This suffices for $M, a \models_{MP} OB$, to complete the lemma.

Theorem 3 follows directly from Lemma 1 and Theorem 2.

Corollary 1. *P is complete with respect to the class of all* (*mono*) *P-models that are reflexive or transitive.*
From the Observation above, if all the relations of $\mathscr{P}_a$ are reflexive or transitive then $\geqslant^*_{\langle a,i\rangle}$ will be reflexive or transitive as well.

Theorem 6. $SDL_aP_e\leq$ *is sound and complete with respect to* (*a*) *the class of all serial MR*$\leqslant$*-models*; *and* (*b*) *the class of all standard MP*$\leqslant$*-models.*

We sketch the proof of (a); (b) follows just as Theorem 2 followed from Theorem 1. To prove (a), we adapt the construction of the canonical model in Theorem 1 to suit the presence of formulas O_aA and also $A \leq B$. Let $M = \langle W, \mathscr{R}, \leqslant, v\rangle$ where W is the class of all maximal consistent extensions of $\mathrm{SDL}_a\mathrm{P}_e\leq$ and $v(p) = \{a : a \in W \text{ and } p \in a\}$, as usual, and $\mathscr{R} = \{R : \exists A(R = R^A)\}$ when, for every formula A, $R^A = \{\langle a, b\rangle : (\text{either } O_eA \notin a \text{ or } A \in b) \text{ and } O_a^{-1}a \subseteq b\}$ with $O_a^{-1}a = \{B : O_aB \in a\}$. For $\leqslant$, define first, for each $a \in W$ and each formula B:

$$\Psi_aB = \{R \in \mathscr{R} : \text{for all } C, \text{ if } \forall b(Rab \rightarrow C \in b), \text{ then } B \leq C \in a\}$$

and then for $R, S \in \mathscr{R}$,

$$R \leqslant_a S \quad \text{iff} \quad \text{for all } C, \text{ if } R \in \Psi_a C, \text{ then there is a } B \text{ such that } S \in \Psi_a B \text{ and } C \leq B \in a$$

Let $\leqslant$ assign each a that relation $\leqslant_a$. It is easy to show that M is an MR$\leqslant$-model, i.e., $\mathscr{R}$ is non-empty and every $R \in \mathscr{R}$ is serial, and $\leqslant_a$ is reflexive and transitive on $\mathscr{R}$. The latter are virtually trivial. Seriality follows from

LITTLE LEMMA A. (*i*) *$O_a^{-1}a$ is consistent*, (*ii*) *If $O_eA \in a$ then $O_a^{-1}a \cup \{A\}$ is consistent*, (*iii*) *if $O_aA \notin a$, then $O_a^{-1}a \cup \{\neg A\}$ is consistent*, (*iv*) *if $O_eA \notin a$, then $O_a^{-1}a \cup \{\neg A\}$ is consistent*

which are easily demonstrated given $\vdash O_aA \rightarrow \neg O_e \neg A$, which follows from $\vdash \neg O_e \bot$ and either $\vdash \neg O_a \bot$ or $\vdash O_aA \rightarrow O_eA$. This too is useful:

LITTLE LEMMA B. (*i*) *If for all b that R^Aab, $B \in b$, then $O_a(A \rightarrow B) \in a$, and hence $A \leq B \in a$; also* (*ii*) *if $O_eA \notin a$ and for all b that R^Aab, $B \in b$, then $O_aB \in a$*

which is also easily proved (compare the proof of Theorem 1). As usual, the theorem follows from

LEMMA 2. *For every A and $a \in W$, $A \in a$ iff $M, a \models_{MP\leqslant} A$.*

Proof: This is proved in the usual way by induction on A. For the case when $A = O_aB$, the argument is much as in standard proofs of completeness for SDL in the Kripke-semantics for serial models. When $A = O_eB$, the argument is as for Theorem 1 above, though Little Lemma A helps. Likewise for the case when $A = B \leq C$. Left-to-right, given an $R^C \in \mathscr{R}$ such that $M, R^C a \models_{MR\leqslant} A$, then R^B will serve for the requisite $S \in \mathscr{R}$. (Show that $M, R^B a \models_{MR\leqslant} B$, and that $R^C \leqslant_a R^B$, applying Little Lemma B and unpacking the definitions of $\leqslant_a$.). Right-to-left, by *reductio*, suppose $M, a \models_{MR\leqslant} B \leq C$ but $B \leq C \notin a$. By the first, for all $R \in \mathscr{R}$ such that $M, Ra \models_{MR\leqslant} B$, there is an $S \in \mathscr{R}$ such that $M, Sa \models_{MR\leqslant} C$ and $R \leqslant_a S$. Show $M, R^B a \models_{MR\leqslant} B$, and take S to be R^D for some D. Since $R^B \leqslant_a R^D$, for any E such that $R^B \in \Psi_a E$, there is an F such that $R^D \in \Psi_a F$ and $E \leq F \in a$. Show that $R^B \in \Psi_a B$, and take F as described. So $B \leq F \in a$. Since $R^D \in \Psi_a F$, i.e., for every G, if $M, R^D a \models_{MR\leqslant} G$, then $F \leq G$, and $M, R^D a \models_{MR\leqslant} C$, $F \leq C \in a$. So, $B \leq C \in a$, by transitivity, contrary to the supposition.

Note that by suppressing all reference to $\leqslant$ in models and to formulas $A \leq B$ we would have a completeness proof for SDL_aP_e (Theorem 4), and hence, $\text{SDL}_a\text{P}_e\leq$ is a conservative extension of SDL_aP_e.

THEOREM 7. *$P_aP_e\leq$ is sound and complete with respect to the class of all MP$\leqslant$-models* (*or the class of all reflexive or transitive MP$\leqslant$-models*).

The proof of this is like that of Theorem 3, but it requires some further complication since $\text{P}_a\text{P}_e\leq$ does not have a multi-relational semantics and so we cannot trade off the proof of a theorem like Theorem 1. Instead we take a detour through a secondary semantics for this system. That is to treat $P_aP_e\leq$ as more traditionally bi-modal, with both operators O_a and O_e being interpreted in the same way but with respect to separate classes of relations. (The binary operator $\leq$ is treated essentially the same as before.)

Thus let a 2MR$\leqslant$-model M be $\langle W, \mathscr{R}^a, \mathscr{R}^e, \leqslant, v\rangle$ with $\mathscr{R}^a$ and $\mathscr{R}^e$ non-empty sets of serial binary relations and $\leqslant$ assigning each $a \in W$ a reflexive, transitive relation $\leqslant_a$ over $\mathscr{R}^e$. Deontic formulas are evaluated according to the rules

$$M, a\vDash_{2MR\leqslant} O_aA \text{ iff } \exists R(R \in \mathscr{R}^a \text{ and } M, Ra\vDash_{2MR\leqslant} A)$$

$$M, a\vDash_{2MR\leqslant} O_eA \text{ iff } \exists R(R \in \mathscr{R}^e \text{ and } M, Ra\vDash_{2MR\leqslant} A)$$

$$M, a\vDash_{2MR\leqslant} A\leq B \text{ iff for every } R \in \mathscr{R}^e \text{ that } M, RA\vDash_{2MR\leqslant} A, \text{ there is an } S \in \mathscr{R}^e \text{ for which } M, Sa\vDash_{2MR\leqslant} B \text{ and } R\leqslant_a S.$$

To validate (O_aO_e), we require $\mathscr{R}^a \subseteq \mathscr{R}^e$, and to validate $(O_a\leq)'$ we require that if $R \in \mathscr{R}^a$ and $S \in \mathscr{R}^e$ then $S\leqslant_a R$.

Lemma 3. *$P_aP_e\leq$ is sound and complete with respect to all such 2MR$\leqslant$-models.*

Proof: Soundness is routine. For completeness, construct the canonical model $M=\langle W, \mathscr{R}^a, \mathscr{R}^e, \leqslant, v\rangle$ with W and v as usual. Let $\mathscr{R}^a = \{R: \exists A(R=R^A_a)\}$ when $R^A_a = \{\langle a, b\rangle: \text{either } O_aA \notin a \text{ or } A \in b\}$ as in Theorem 1. Let $\mathscr{R}^{e*} = \{R: \exists A(R=R^A_e)\}$ when $R^A_e = \{\langle a, b\rangle: \text{either } O_eA \notin a$ or $A \in b\}$, and let $\mathscr{R}^e = \mathscr{R}^a \cup \mathscr{R}^{e*}$. (Hence, trivially, $\mathscr{R}^a \subseteq \mathscr{R}^e$). Define $\leqslant$ just as for Theorem 6, with the same definitions of Ψ_a and $\leqslant_a$ for all pairs of relations in $\mathscr{R}^e$. It is easy to show that M is a 2MR$\leqslant$-model. It is also fairly routine to demonstrate that for all A, $A \in a$ iff $M, a\vDash_{2MR\leqslant} A$. Here these Little Lemmas are useful:

Little Lemma C. (*i*) *If for all b that R^Aab, $B \in b$, then $\vdash A\to B$, and also* (*ii*) *if $O_eA \notin a$ and for all b that R^Aab, $B \in b$, then $\vdash B$.*

Little Lemma D. (*i*) *If $R=R^A$ then $R \in \Psi_aA$,* (*ii*) *if $R \in \mathscr{R}^a$ and $O_aA \notin a$ then $R \in \Psi_a\top$, and* (*iii*) *if $R \in \mathscr{R}^e$ and $O_eA \notin a$ then $R \in \Psi_a\top$.*

From this construction, completeness follows as usual.

We may similarly define a bi-modal 2MP$\leqslant$ semantics for $P_aP_e\leq$ with models $\langle W, \mathscr{P}^a, \mathscr{P}^e, \leqslant, v\rangle$ in the obvious way, and then prove

Lemma 4. *$P_aP_e\leq$ is sound and complete with respect to all such 2MP$\leqslant$-models.* from Lemma 3 just as Theorem 2 followed from Theorem 1. 2MP$\leqslant$-models may be reflexive, transitive, or connected just as their regular non-bi-modal MP$\leqslant$ counterparts are, and the proof of Lemma 4 makes it apparent that $P_aP_e\leq$ is sound and complete with respect to all these variations, just as under Theorem 2.

We can now prove Theorem 7 with a construction similar to that for Theorem 3. Given a 2MP$\leqslant$-model $M = \langle W, \mathscr{P}^a, \mathscr{P}^e, \leqslant, v\rangle$ with every relation in $\mathscr{P}^e_a$ reflexive on its field, let $M^{\#} = \langle W^{\#}, \mathscr{P}^{\#}, \leqslant^{\#}, v^{\#}\rangle$, be defined thus: As earlier, let every relation $\geqslant_a \in \mathscr{P}^e_a$ be assigned a unique index, i, and let I_a be the set of all such indexes. Let I be the union of all such I_a for $a \in W$. Let $W^{\#}$ be the set of all pairs $\langle a, i\rangle$ such that $a \in W$ and $i \in I$. For each relation $\geqslant^j_a \in \mathscr{P}^e_a$, define a relation on $W^{\#}$, $\geqslant^{*j}_{\langle a,i\rangle}$, such that for each $a \in W$ and each $i \in I$

$$\geqslant^{*j}_{\langle a,i\rangle} = \{\langle\langle b, k\rangle, \langle c, l\rangle\rangle : k = l = j \text{ and } b \leqslant^j_a c\}$$

Next, define a relation

$$Q_{\langle a,i\rangle} = \bigcup\{\geqslant^{*j}_{\langle a,i\rangle} : \geqslant^j_a \in \mathscr{P}^a_a\}$$

and then

$$\geqslant^{\#j}_{\langle a,i\rangle} = \geqslant^{*j}_{\langle a,i\rangle} \cup Q_{\langle a,i\rangle}$$

Let $\mathscr{P}^{\#}_{\langle a,i\rangle}$ be the set of all such $\geqslant^{\#j}_{\langle a,i\rangle}$ for all $j \in I_a$, and let $\mathscr{P}^{\#}$ assign $\mathscr{P}^{\#}_{\langle a,i\rangle}$ to $\langle a, i\rangle$. Further, for $\geqslant^{\#j}_{\langle a,i\rangle}, \geqslant^{\#k}_{\langle a,i\rangle} \in \mathscr{P}^{\#}_{\langle a,i\rangle}$, let $\geqslant^{\#j}_{\langle a,i\rangle} \leqslant^{\#}_{\langle a,i\rangle} \geqslant^{\#k}_{\langle a,i\rangle}$ if $\geqslant^j_a \leqslant_a \geqslant^k_a$. $\leqslant^{\#}$ assigns the relation $\leqslant^{\#}_{\langle a,i\rangle}$ to $\langle a, i\rangle$. Finally, $v^{\#}(p) = \{\langle a, i\rangle : a \in v(p) \text{ and } i \in I\}$. $M^{\#}$ is plainly a MP$\leqslant$ model. Further

LEMMA 5. *For all $i \in I$, $M, a \vDash_{2MP\leqslant} A$ iff $M^{\#}, \langle a, i\rangle \vDash_{MP\leqslant} A$.*

Proof: As usual, by induction on A. We consider the deontic cases. Case (a) $A = O_a B$. (i) Suppose $M, a \vDash_{2MP\leqslant} O_a B$ i.e., there is a $\geqslant^j_a \in \mathscr{P}^a$, such that $M, \geqslant^j_a a \vDash_{2MP\leqslant} B$. Hence there is a $b \in \mathscr{F}\geqslant^j_a$ such that $M, b \vDash_{2MP\leqslant} B$ and for all c that $c \geqslant^j_a b$, $M, c \vDash_{2MP\leqslant} B$. Suppose then that $\geqslant^{\#k}_{\langle a,i\rangle}$ is some relation in $\mathscr{P}^{\#}_{\langle a,i\rangle}$ and consider $\langle b, j\rangle$. Since $b \in \mathscr{F}\geqslant^j_a$, $\langle b, j\rangle \in \mathscr{F}\geqslant^{\#j}_{\langle a,i\rangle}$. Further, $M^{\#}, \langle b, j\rangle \vDash_{MP\leqslant} B$ by the inductive hypothesis. Next, consider any $c^{\#}$ that $c^{\#} \geqslant^{\#k}_{\langle a,i\rangle} \langle b, j\rangle$. Since $\langle c^{\#}, \langle b, j\rangle\rangle \in \geqslant^{\#k}_{\langle a,i\rangle}$, $\langle c^{\#}, \langle b, j\rangle\rangle \in \geqslant^{*k}_{\langle a,i\rangle}$ or $\langle c^{\#}, \langle b, j\rangle\rangle \in Q_{\langle a,i\rangle}$. In the first case $c^{\#} = \langle c, j\rangle$ and $k = j$ and $c \geqslant^j_a b$. So $M, c \vDash_{MP\leqslant} B$, and then $M^{\#}, \langle c, j\rangle \vDash_{MP\leqslant} B$ by the inductive hypothesis, thus $M^{\#}, c^{\#} \vDash_{MP\leqslant} B$. In the latter case, there is a $\geqslant^{*l}_{\langle a,i\rangle}$ where $\geqslant^l_a \in \mathscr{P}^a_a$ and $\langle c^{\#}, \langle b, j\rangle\rangle \in \geqslant^{*l}_{\langle a,i\rangle}$, but then again $j = l$ and $c \geqslant^l_a b$, i.e., $c \geqslant^j_a b$. So again $M, c \vDash_{2MP\leqslant} B$, and $M^{\#}, \langle c, j\rangle \vDash_{MP\leqslant} B$ by the inductive hypothesis, and since $c^{\#} = \langle c, j\rangle$, $M^{\#}, c^{\#} \vDash_{MP\leqslant} B$. Thus, in either case $M^{\#}, c^{\#} \vDash_{MP\leqslant} B$, which suffices for $M^{\#}, \langle a, i\rangle \vDash_{MP\leqslant} O_a B$. (ii) Suppose $M^{\#}, \langle a, i\rangle \vDash_{MP\leqslant} O_a B$, i.e., for every $\geqslant^{\#j}_{\langle a,i\rangle} \in \mathscr{P}^{\#}_{\langle a,i\rangle}$, $M^{\#}, \geqslant^{\#j}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} B$. Since $\mathscr{P}^a_a$ is not empty, let $\geqslant^k_a \in \mathscr{P}^a_a$; plainly $\geqslant^{\#k}_{\langle a,i\rangle} \in \mathscr{P}^{\#}_{\langle a,i\rangle}$, so $M^{\#}, \geqslant^{\#k}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} B$, i.e., there is a $b^{\#} \in \mathscr{F}\geqslant^{\#k}_{\langle a,i\rangle}$ such that $M^{\#}, b^{\#} \vDash_{MP\leqslant} B$ and for every $c^{\#}$ with $c^{\#} \geqslant^{\#k}_{\langle a,i\rangle} b^{\#}$, $M^{\#}, c^{\#} \vDash_{MP\leqslant} B$. With $b^{\#} \in \mathscr{F}\geqslant^{\#k}_{\langle a,i\rangle}$, either $b^{\#} \in \mathscr{F}\geqslant^{\#k}_{\langle a,i\rangle}$ or $b^{\#} \in \mathscr{F}Q_{\langle a,i\rangle}$. In the first case $b^{\#} = \langle b, k\rangle$ for some $b \in W$ and $k \in I$, and then $b \in \mathscr{F}\geqslant^k_a$ and also $M, b \vDash_{2MP\leqslant} B$ by the inductive hypothesis. If c is any point that $c \geqslant^k_a b$, then $\langle c, k\rangle \geqslant^{*k}_{\langle a,i\rangle} \langle b, k\rangle$; so $M^{\#}, \langle c, k\rangle \vDash_{MP\leqslant} B$, and $M, c \vDash_{2MP\leqslant} B$ by the inductive hypothesis, which would suffice for $M, a \vDash_{2MP\leqslant} O_a B$. In the other case, there is a $\geqslant^{*l}_{\langle a,i\rangle}$ where $\geqslant^l_a \in \mathscr{P}^a_a$ and $b^{\#} = \langle b, l\rangle$. The argument proceeds as before that $M, a \vDash_{2MP\leqslant} O_a B$, which completes this case.

Case (b), when $A = O_e B$, is similar, though easier, and may be left to the reader.

Case (c) $A = B \leq C$. (i) Suppose $M, a \vDash_{2MP\leqslant} B \leq C$, so that for every $R \in \mathscr{R}^e$ such that $M, Ra \vDash_{2MR\leqslant} B$, there is an $S \in \mathscr{R}^e$ for which $M, Sa \vDash_{2MR\leqslant} C$ and $R \leqslant_a S$. Now consider any arbitrary $\geqslant^{\#j}_{\langle a,i\rangle} \in \mathscr{P}^{\#}_{\langle a,i\rangle}$ that $M^{\#}, \geqslant^{\#j}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} B$. $\geqslant^j_a \in \mathscr{P}^e_a$. Since $M^{\#}, \geqslant^{\#j}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} B$, there is a $b^{\#} \in \mathscr{F} \geqslant^{\#j}_{\langle a,i\rangle}$ such that $M^{\#}, b^{\#} \vDash_{MP\leqslant} B$ and for all $c^{\#}$ if $c^{\#} \geqslant^{\#j}_{\langle a,i\rangle} b^{\#}$ then $M^{\#}, c^{\#} \vDash_{MP\leqslant} B$. Either (1) $b^{\#} \in \mathscr{F} \geqslant^{*j}_{\langle a,i\rangle}$ or (2) $b^{\#} \in \mathscr{F} Q_{\langle a,i\rangle}$. In case (1), $b^{\#} = \langle b, j\rangle$ for some $b \in W$ and $j \in I$ and $M^{\#}, \langle b, j\rangle \vDash_{MP\leqslant} B$, so $M, b \vDash_{2MP\leqslant} B$ by the inductive hypothesis. Let $c \in W$ be any point that $c \geqslant^j_a b$; then $\langle c, j\rangle \geqslant^{*j}_{\langle a,i\rangle} \langle b, j\rangle$ and so $\langle c, j\rangle \geqslant^{\#j}_{\langle a,i\rangle} \langle b, j\rangle$, in which case $M^{\#}, \langle c, j\rangle \vDash_{MP\leqslant} B$ and so $M, c \vDash_{2MP\leqslant} B$ by the inductive hypothesis. That shows that $M, \geqslant^j_a a \vDash_{2MP\leqslant} B$. Hence there is a $\geqslant^k_a \in \mathscr{P}^e_a$ for which $M, \geqslant^k_a a \vDash_{2MP\leqslant} C$ and $\geqslant^j_a \leqslant_a \geqslant^k_a$. $\geqslant^{\#k}_{\langle a,i\rangle} \in \mathscr{P}^{\#}_{\langle a,i\rangle}$. We show that $M^{\#}, \geqslant^{\#k}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} C$ and that $\geqslant^{\#j}_{\langle a,i\rangle} \leqslant_a \geqslant^{\#k}_{\langle a,i\rangle}$. For the first, since there is a $c \in \mathscr{F} \geqslant^k_a$ and $M, c \vDash_{2MP\leqslant} C$, $\langle c, k\rangle \in \mathscr{F} \geqslant^{*k}_{\langle a,i\rangle}$ and so $\langle c, k\rangle \in \mathscr{F} \geqslant^{\#k}_{\langle a,i\rangle}$. Also $M^{\#}, \langle c, k\rangle \vDash_{MP\leqslant} C$ by the inductive hypothesis. Let $d^{\#}$ be any point such that $d^{\#} \geqslant^{\#k}_{\langle a,i\rangle} \langle c, k\rangle$. Either $d^{\#} \geqslant^{*k}_{\langle a,i\rangle} \langle c, k\rangle$ or $d^{\#} Q_{\langle a,i\rangle} \langle c, k\rangle$. In the first case, $d^{\#} = \langle d, k\rangle$ and $d \geqslant^k_a c$, and so $M, d \vDash_{2MP\leqslant} C$ and then $M^{\#}, \langle d, k\rangle \vDash_{MP\leqslant} C$ by the inductive hypothesis, and therefore $M^{\#}, \geqslant^{\#k}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} C$. In the other case, if $d^{\#} Q_{\langle a,i\rangle} \langle c, k\rangle$, then there is a $\geqslant^{*m}_{\langle a,i\rangle}$ such that $d^{\#} \geqslant^{*m}_{\langle a,i\rangle} \langle c, k\rangle$ when $\geqslant^m_a \in \mathscr{P}^a_a$. But then, if $d^{\#} = \langle d, n\rangle$ for some index n, $n = m = k$ and so $\langle d, k\rangle \geqslant^{*k}_{\langle a,i\rangle} \langle c, k\rangle$ and $d \geqslant^k_a c$, as before, whence $M, d \vDash_{2MP\leqslant} C$, and then $M^{\#}, \langle d, k\rangle \vDash_{MP\leqslant} C$ by the inductive hypothesis, and so $M^{\#}, \geqslant^{\#k}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} C$. That $\geqslant^{\#j}_{\langle a,i\rangle} \leqslant^{\#}_{\langle a,i\rangle} \geqslant^{\#k}_{\langle a,i\rangle}$ follows from $\geqslant^j_a \leqslant_a \geqslant^k_a$, which suffices for $M^{\#}, \langle a, i\rangle \vDash_{MP\leqslant} B \leq C$ in this case. For the other case, (2), when $b^{\#} \in \mathscr{F} Q_{\langle a,i\rangle}$ the argument is similar.

For (ii), suppose $M^{\#}, \langle a, i\rangle \vDash_{MP\leqslant} B \leq C$, so that for every relation $\geqslant^{\#j}_{\langle a,i\rangle} \in \mathscr{P}^{\#}_{\langle a,i\rangle}$ that $M^{\#}, \geqslant^{\#j}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} B$, there is a relation $\geqslant^{\#k}_{\langle a,i\rangle} \in \mathscr{P}^{\#}_{\langle a,i\rangle}$ such that $M^{\#}, \geqslant^{\#k}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} C$ and $\geqslant^{\#j}_{\langle a,i\rangle} \leqslant^{\#}_{\langle a,i\rangle} \geqslant^{\#k}_{\langle a,i\rangle}$. To show $M, a \vDash_{2MP\leqslant} B \leq C$, let $\geqslant^j_a \in \mathscr{P}^e_a$ be any relation that $M, \geqslant^j_a a \vDash_{2MP\leqslant} B$. Argue as before that $M^{\#}, \geqslant^{\#j}_{\langle a,i\rangle} \langle a, i\rangle \vDash_{MP\leqslant} B$. Then there is a relation $\geqslant^{\#k}_{\langle a,i\rangle} \in \mathscr{P}^{\#}_{\langle a,i\rangle}$ such that $M^{\#}, \geqslant^{\#k}_{\langle a,i\rangle} \vDash_{MP\leqslant} C$ and $\geqslant^{\#j}_{\langle a,i\rangle} \leqslant^{\#}_{\langle a,i\rangle} \geqslant^{\#k}_{\langle a,i\rangle}$. Argue as before that $M, \geqslant^k_a \vDash_{2MP\leqslant} C$. Further, since $\geqslant^{\#j}_{\langle a,i\rangle} \leqslant^{\#}_{\langle a,i\rangle} \geqslant^{\#k}_{\langle a,i\rangle}$, $\geqslant^j_a \leqslant_a \geqslant^k_a$ by definition. That suffices to show that $M, a \vDash_{2MP\leqslant} B \leq C$, as required. This completes the Lemma.

Theorem 7 now follows from Lemmas 4 and 5 in the usual way. Note again that this contains a proof of completeness for P_aP_e (Theorem 5); thus, $P_aP_e\leq$ is a conservative extension of P_aP_e.[9]

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