NORMATIVE SYSTEMS REPRESENTED BY BOOLEAN QUASI-ORDERINGS

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1. Introduction

By a normative system jurists often mean the law of a country, like Swedish law, or part of the law of a country, such as the Swedish law of contracts. With regard to the issue how a logical reconstruction of a normative system should be made, a central question is what kind of entities a normative system is composed of and how it should be represented. The aim of the present paper is to contribute to the study of this field. More specifically, our contribution aims at presenting a framework for analysing different kinds of joinings of conceptual structures in a normative system. A case in view is where one of these structures is descriptive and the other normative.

In a previous paper, we presented a working model where conceptual structures were represented by lattices.[1] This was the first step in developing the theory, having the limitation that the role of negation for concept formation was not dealt with. This limitation is eliminated in the present paper, which is different in several respects. The framework to be developed is based on the theory of Boolean algebra instead of lattice theory. The basic kind of relations dealt with are quasi-orderings rather than partial orderings, as was the case in our previous paper, where partial orderings were introduced by a transition to equivalence classes. The framework of what we will call "Boolean quasi-orderings" is general in the sense that the main results are not tied to a specific interpretation in terms of conditions as was the case in the earlier paper. Thus, the case where the domains of the orderings have conditions as their members, so-called condition implication structures, only plays the part of one of several models of the theory. Also, the new framework is more flexible in the sense that it is not confined to the joining of two subsystems within a background system. In the new framework it is possible to generate a background system when subsystems are given.

The paper is intended as an overview of some central ideas and results pertaining to the theory as it is now developed. For this reason we do not

[1] See Lindahl and Odelstad 1999.

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give the proofs of the results stated, and the results are stated in the text without numbering of theorems and lemmas.

2. Boolean Quasi-orderings as Representing Normative Systems

2.1. The Notion of a Boolean Quasi-ordering

The central notion in what follows is the notion of a Boolean quasi-ordering.

Definition 1. A structure $\langle B, \wedge, ', R\rangle$ is a *Boolean quasi-ordering* if $\langle B, \wedge, '\rangle$ is a Boolean algebra with $\top$ as the unit element and $\bot$ as the zero element and R is a quasi-ordering on B (i.e., a reflexive and transitive relation on B), which satisfies the following conditions for all a, b and c in B:

(1) aRb and aRc implies $aR(b \wedge c)$.
(2) aRb implies $b'Ra'$.
(3) $(a \wedge b)Ra$.
(4) not $\top R \bot$.[2]

The indifference part of R is denoted Q and is defined by: aQb if and only if aRb and bRa. Similarly, the strict part of R is denoted S and is defined by: aSb if and only if aRb and not bRa.

Let $\leqslant$ be the partial ordering determined by $\langle B, \wedge, '\rangle$ where $\leqslant$ is defined by $a \leqslant b$ if and only if $a \wedge b = a$. We note that from (3) it follows that $a \leqslant b$ implies aRb. For, by the definition, $a \leqslant b$ implies $a \wedge b = a$ and, since $(a \wedge b)Rb$ it follows that aRb.

2.2. Condition Implication Structures as Models of Boolean Quasi-orderings

The theory of Boolean quasi-orderings has many models. Primarily, we are interested in a kind of models called *condition implication structures* (abbreviated "*cis*"). Let $M = \{a, b, c, \ldots\}$ be a set of conditions, where

a' is the condition defined by $a'(x_1, \ldots, x_\nu)$ if and only if not $a(x_1, \ldots, x_\nu)$,
$a \wedge b$ is defined by $(a \wedge b)(x_1, \ldots, x_\nu)$ if and only if $a(x_1, \ldots, x_\nu)$ and $b(x_1, \ldots, x_\nu)$,
$a \vee b$ is defined by $(a \vee b)(x_1, \ldots, x_\nu)$ if and only if $a(x_1, \ldots, x_\nu)$ or $b(x_1, \ldots, x_\nu)$.

[2] In Odelstad and Lindahl 1998 a structure $\langle B, \wedge, ', R\rangle$ fulfilling (1) and (2) was called a Boolean quasi-ordering, while if (3) and (4) are fulfilled as well, the structure was called a *regular* Boolean quasi-ordering. For some purposes, related to the idea of "defeasibility", it can be desirable to drop (3). (We note that from (3) it follows that if aRb, then $(a \wedge c)Rb$.) In the present paper, however, we do not develop this idea, and, for convenience, a structure fulfilling (1)–(4) is called, simply, a Boolean quasi-ordering.

The ν-ary empty condition is the condition $\bot$ such that for no $x_1, \ldots, x_\nu$, $\bot(x_1, \ldots, x_\nu)$. The ν-ary universal condition is the condition $\top$ such that for all $x_1, \ldots, x_\nu$, $\top(x_1, \ldots, x_\nu)$.

The procedure of forming compounds can be iterated. For example, if a is the condition "to be a woman", b is the condition "to be a parent of", and c is the condition "to be an adoptive parent of", then $(a \wedge b) \vee c$ is the condition of being mother or adoptive parent of.

Let M^* be the closure of M under the operations $\wedge$, $'$, and let $\mathscr{S}$ be a normative system. The model *cis* we have in view is a Boolean quasi-ordering $\langle M^*, \wedge, ', R\rangle$ such that aRb represents that it follows from $\mathscr{S}$ that a implies b. For example, if a is the condition of being less than fifteen years old and b is the condition of being liable to punishment, the statement that from $\mathscr{S}$ it follows that a implies b', is represented by aRb'. Note that if $\leqslant$ is the partial ordering defined by $a \leqslant b$ if and only if $a \wedge b = a$, then $\leqslant$ expresses implication due to logic alone.

The important function of a normative system is to express what can be called "normative correlations".[3] If a *cis* represents the normative system $\mathscr{S}$, we will say that aRb describes a normative correlation for $\mathscr{S}$ if a is a descriptive and b is a normative condition.

A *cis* can be applied to individuals by deductions of the following kind, where i, j are individual constants, for instance names of individuals. Let $\langle M^*, \wedge, ', R\rangle$ be a *cis* representing the normative system $\mathscr{S}$. Then we will say that $b(i, j)$ is deducible according to $\mathscr{S}$ from $a(i, j)$ if it holds that aRb.

Conditions, as appearing in a *cis* have many affinities with relations, if, as is usual, relations are regarded extensionally as sets of ordered n-tuples. Perhaps it is appropriate to say that a theory of conditions in the way it is developed here is a modified theory of relations.[4] Obviously, the operations of negation, conjunction and disjunction for conditions have as counterparts the operations of complement, intersection and union for relations. However, if R_1 and R_2 are relations of different arity, their intersection $R_1 \cap R_2$ is empty and their union is not a relation. For example the intersection between a set of pairs and a set of triples is empty, and the union of a set of pairs and a set of triples is not a relation. The case is different with conditions, since in forming conjunctions and disjunctions, we adopt the rule that the arity of $a \wedge b$ and $a \vee b$, respectively, equals the greatest of the arities of a and b.[5] The arity of a' is always the same as the arity of a.

[3] Cf. Alchourrón and Bulygin 1971, p. 55.

[4] In our approach of considering Boolean algebras of conditions, we have been inspired by some lectures of Stig Kanger's, given in the Fall of 1977. In these lectures, Kanger started developing an algebraic theory of conditions, based on Boolean and cylindric algebras.

[5] This means that if a is μ-ary and b is ν-ary and $\phi = \max\{\mu, \nu\}$, then, for all $x_1, \ldots, x_\phi$, $(a \wedge b)(x_1, \ldots, x_\phi)$ if and only if $a(x_1, \ldots, x_\mu)$ and $b(x_1, \ldots, x_\nu)$.

2.3. Boolean Quasi-orderings, Congruence Relations and Homomorphisms

We will now take a closer look at the kind of structures we call Boolean quasi-orderings. There is a close relationship between Boolean quasi-orderings and proper congruence relations.[6] If $\langle B, \wedge, ', R\rangle$ is a Boolean quasi-ordering, then Q (i.e., the indifference part of R) is a proper congruence relation on the Boolean algebra $\langle B, \wedge, '\rangle$. And if Θ is a proper congruence relation, then Θ determines a Boolean quasi-ordering, namely the relation R on B defined by aRb iff $a \wedge b \Theta a$. (Note that $Q = \Theta$.)

Next, suppose that $\langle B, \wedge, ', R\rangle$ is a Boolean quasi-ordering and that Q is the indifference part of R. Since Q is a proper congruence relation on the Boolean algebra $\langle B, \wedge, '\rangle$, we can define a binary operation on B/Q in the following way, where $[c]$ denotes the equivalence class with respect to Q generated by c, and $B/Q = \{[c] \mid c \in B\}$:

$$[a] \cap [b] = [a \wedge b]$$

Furthermore, we can define a unary operation on B/Q as follows:

$$-[a] = [a'].$$

As is well-known $\langle B/Q, \cap, -\rangle$ *is a Boolean algebra and the mapping*

$$p: B \to B/Q, \text{ where } p(b) = [b]$$

is a homomorphism of $\langle B, \wedge, '\rangle$ onto $\langle B/Q, \cap, -\rangle$.[7] ($p$ is often called the natural mapping on B onto B/Q, and $\langle B/Q, \cap, -\rangle$ is called the quotient algebra of $\langle B, \wedge, '\rangle$ with respect to Q.)

The set $I = \{a \in B \mid aQ\bot\}$, as is well known, is a proper ideal and aQb iff $a + b \in I$, where $+$ is the symmetric difference, i.e. $a + b = (a \wedge b') \vee (a' \wedge b)$. I is the kernel of the natural homomorphism p and consists of the elements in B which are mapped by p onto the zero element in $\langle B/Q, \cap, -\rangle$.[8] Note that aRb iff $(a \wedge b)\ Qa$ iff $(a \wedge b) + a \in I$ iff $(a \wedge b') \in I$.

It appears that there are two Boolean algebras which should be kept apart. First, there is the Boolean algebra $\langle B, \wedge, '\rangle$ the domain B of which is the field of the relation R. After adding the relation R we get the Boolean

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[6] We remind the reader of the notion of a proper congruence relation. Θ is a congruence relation on the Boolean algebra $\langle B, \wedge, '\rangle$ if Θ is an equivalence relation on B and the following two conditions holds:

(i) $a\Theta b$ implies $a \wedge c \Theta b \wedge c$ for all $c \in B$
(ii) $a\Theta b$ implies $b' \Theta a'$.

Θ is a proper congruence relation on $\langle B, \wedge, '\rangle$ if Θ is a congruence relation which is different from the universal relation on B.

[7] See for example Stoll 1963, pp. 259 ff.

[8] Ibid.

quasi-ordering $\langle B, \wedge, ', R\rangle$. By switching to the equivalence classes defined by the indifference part Q of R, we get another Boolean algebra, i.e., the quotient algebra $\langle B/Q, \cap, -\rangle$ with respect to Q.

Although, by a transition to equivalence classes, from a Boolean quasi-ordering we get a new Boolean algebra, in what follows we will not make this transition. The point is that, in a *cis*, for two conditions a and b, although it holds that aQb, and therefore a and b belong to the same Q-equivalence class, we may want to distinguish a and b because they may have different meaning. Therefore, there is a point in remaining within the framework of Boolean quasi-orderings as defined above.

3. The Joining of Fragments of a Boolean Quasi-ordering

3.1. Fragments, Joinings, and Connections

As suggested in the introduction, our paper aims at presenting a framework for analysing different kinds of joinings of conceptual structures in a normative system. For this reason a central part of our inquiry is the distinction between various parts of a Boolean quasi-ordering and the different ways such parts can be combined. We therefore proceed to the definition of a fragment $\langle B_1, \wedge, ', R_1\rangle$ of a Boolean quasi-ordering $\langle B, \wedge, ', R\rangle$.

Definition 2. If $\langle B, \wedge, ', R\rangle$ is a Boolean quasi-ordering, and $\langle B_1, \wedge, '\rangle$ is a subalgebra of $\langle B, \wedge, '\rangle$, and $R_1 = R/B_1$, then the structure $\langle B_1, \wedge, ', R_1\rangle$ will be called a *fragment* of $\langle B, \wedge, ', R\rangle$.[9]

(The expression R/B_1 denotes the restriction of the relation R to the set B_1.) To simplify the notation, in the sequel we will use the convention that Boolean quasi-orderings $\langle B, \wedge, ', R\rangle$ and $\langle B_i, \wedge, ', R_i\rangle$ are denoted simply by $\mathscr{B}$ and $\mathscr{B}_i$ respectively.

We note that if $\mathscr{B}$ is a Boolean quasi-ordering and $\mathscr{B}_1$ is a fragment of $\mathscr{B}$, then $\mathscr{B}_1$ is a Boolean quasi-ordering.

The most general notion we use for the tying of two fragments is that of a joining.

Definition 3. A *joining* from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$ is a pair $\langle b_1, b_2\rangle$ in $\mathscr{B}$ such that $b_1 \in B_1$, $b_2 \in B_2$, $b_1 R b_2$, not $b_1 R \bot$ and not $\top R b_2$. A joining $\langle b_1, b_2\rangle$ from $\mathscr{B}_1$ to $\mathscr{B}_2$ is called *strict* if $b_1 S b_2$.

[9] We remind the reader of the notion of a subalgebra. If $\langle B, \wedge, '\rangle$ is a Boolean algebra and A is a non-empty subset of B such that A is closed under the operations $\wedge$ and $'$, then $\langle A, \wedge_A, '_A\rangle$ is a subalgebra of $\langle B, \wedge, '\rangle$ where $\wedge_A$ and $'_A$ are restrictions of the operations $\wedge$ and $'$ to A. (That A is closed under the operations $\wedge$ and $'$ means that if $a, b \in A$ then $a \wedge_A b \in A$ and $a'_A \in A$.)

A central kind of joinings are those called *connections*.

DEFINITION 4. Suppose that $\mathscr{B}$ is a Boolean quasi-ordering and $\mathscr{B}_1$ and $\mathscr{B}_2$ are fragments of $\mathscr{B}$. Then $\langle b_1, b_2\rangle$ is a *connection* from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$ if the following four requirements are satisfied:

(i) $\langle b_1, b_2\rangle$ is a joining from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$.
(ii) There is $a_1 \in B_1\backslash B_2$ and $a_2 \in B_2\backslash B_1$ such that a_1Rb_1 and b_2Ra_2.
(iii) If $a_1 \in B_1$ and $b_1Ra_1Rb_2$ then a_1Rb_1.
(iv) If $a_2 \in B_2$ and $b_1Ra_2Rb_2$ then b_2Ra_2.

((iii)–(iv) are called the proximity principles.) Intuitively, if $\langle b_1, b_2\rangle$ is a connection, then there is no element in B_1 or B_2 which is "strictly between" b_1 and b_2.

We note that if $\langle b_1, b_2\rangle$ is a connection from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$, then $\langle b_2', b_1'\rangle$ is a connection from $\mathscr{B}_2$ to $\mathscr{B}_1$ in $\mathscr{B}$. We call $\langle b_2', b_1'\rangle$ the *converse* of the connection $\langle b_1, b_2\rangle$.

Suppose $\mathscr{B}$ is a finite Boolean quasi-ordering, and that $\mathscr{B}_1$ and $\mathscr{B}_2$ are fragments of $\mathscr{B}$. Then it can be proved that if there is $a_1 \in B_1\backslash B_2$ and $a_2 \in B_2\backslash B_1$ such that a_1Ra_2, then there is a connection $\langle b_1, b_2\rangle$ from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$ such that a_1Rb_1 and b_2Ra_2. Informally, in a finite Boolean quasi-ordering, each joining such that $b_1 \in B_1\backslash B_2$ and $b_2 \in B_2\backslash B_1$ encompasses a connection.

To clarify why the notion of a connection is important, let us think of two fragments $\mathscr{B}_1$ and $\mathscr{B}_2$, which are of different kinds. For example, we can assume that $\mathscr{B}_1$ has a domain of descriptive conditions while the domain of $\mathscr{B}_2$ is normative. Suppose that we are interested in how, by means of an implicative relation R, these two fragments can be combined in the larger Boolean quasi-ordering $\mathscr{B}$. If the first fragment is descriptive and the second is normative, a connection can be thought of as representing the specific legal content, for example the result of legal enactment. This is especially plausible if R_1 equals $\leqslant$ restricted to B_1 and R_2 equals $\leqslant$ restricted to B_2, i.e., if R, restricted to B_1 and B_2 respectively, is implication due to logic alone.

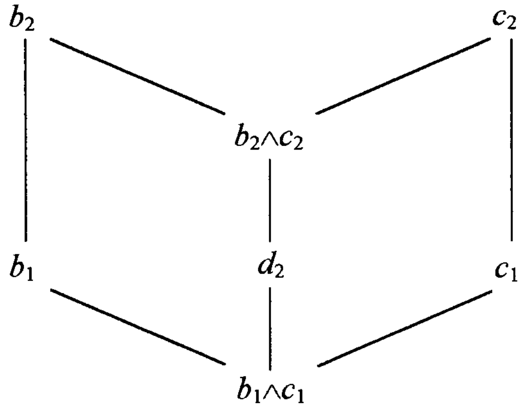
3.2. A Closer Look at the Set of Connections

We shall now take a look at the set of connections from one fragment to another and say a little about the structure of such a family.

Firstly, we observe that if $\langle b_1, b_2\rangle$ and $\langle c_1, c_2\rangle$ are connections from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$, then $b_1R_1c_1$ if and only if $b_2R_2c_2$. Secondly, given two connections $\langle b_1, b_2\rangle$ and $\langle c_1, c_2\rangle$ from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$, where $\mathscr{B}$ is finite, we can form new connections by using conjunction and disjunction, namely as follows:

(1) If $\langle b_1 \wedge c_1, b_2 \wedge c_2)$ is a joining, then there is a $d_2 \in B_2$ such that $\langle b_1 \wedge c_1, d_2 \rangle$ is a connection, and $d_2 R_2 (b_2 \wedge c_2)$.
(2) If $\langle b_1 \vee c_1, b_2 \vee c_2 \rangle$ is a joining, then there is a $d_1 \in B_1$ such that $\langle d_1, b_2 \vee c_2 \rangle$ is a connection, and $(b_1 \vee c_1) R d_1$.

As a remark to case (1), let us consider the following illustration:



If there are two connections $\langle b_1, b_2 \rangle$, $\langle c_1, c_2 \rangle$ it can be the case that for $d_2 \in B_2$, $\langle b_1 \wedge c_1, d_2 \rangle$ is a connection from $\mathscr{B}_1$ to $\mathscr{B}_2$, where $d_2 S(b_2 \wedge c_2)$. This seems to be relevant for a discussion of "organic wholes".

With a view to the classification of different kinds of connections, it is pertinent to introduce the following relation on pairs of elements in B.

Definition 5. Let $\mathscr{B}$ be a Boolean quasi-ordering and $a, b, c, d \in B$. We say that the pair $\langle a, b \rangle$ is *at least as low as* $\langle c, d \rangle$ if aRc and bRd. We use "$\underline{\ll}$" to denote "at least as low as". Thus

$$\langle a, b \rangle \underline{\ll} \langle c, d \rangle \text{ if and only if } aRc \text{ and } bRd.$$

The equality part of $\underline{\ll}$, denoted $\approx$ (and expressing similarity), is defined by $\langle a, b \rangle \approx \langle c, d \rangle$ if and only if aQc and bQd.

The relation "at least as low as" is a quasi-ordering, i.e. transitive and reflexive. The number of equivalence classes "up to similarity" is of interest with respect to the notion of "coupling" to be introduced in the next section.

3.3. Couplings and Pair Couplings

Next we introduce the special case of a connection called a *coupling*.

Definition 6. Suppose that $\mathscr{B}$ is a Boolean quasi-ordering and $\mathscr{B}_1$ and $\mathscr{B}_2$ are fragments of $\mathscr{B}$. Then $\langle b_1, b_2 \rangle$ is a *coupling* from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$ if the following three requirements are satisfied:

(i) $\langle b_1, b_2 \rangle$ is a joining from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$.
(ii) There is $a_1 \in B_1 \backslash B_2$ and $a_2 \in B_2 \backslash B_1$ such that $a_1 R b_1$ and $b_2 R a_2$.
(iii) If $a_1 \in B_1$, $a_2 \in B_2$ and $a_1 R a_2$, then $a_1 R b_1$ and $b_2 R a_2$.

It is easy to see that every coupling from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$ is also a connection. It is also possible to prove that if $\langle b_1, b_2 \rangle$ is a coupling and $\langle c_1, c_2 \rangle$ a connection from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$, where $\mathscr{B}$ is finite, then $\langle b_1, b_2 \rangle \approx \langle c_1, c_2 \rangle$. This implies that if there are more couplings than one from $\mathscr{B}_1$ to $\mathscr{B}_2$, these couplings are similar.

A notion of some interest is that of a pair coupling. The definition goes as follows.

DEFINITION 7. The set $\{\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle\}$ is a *pair coupling* from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$ if

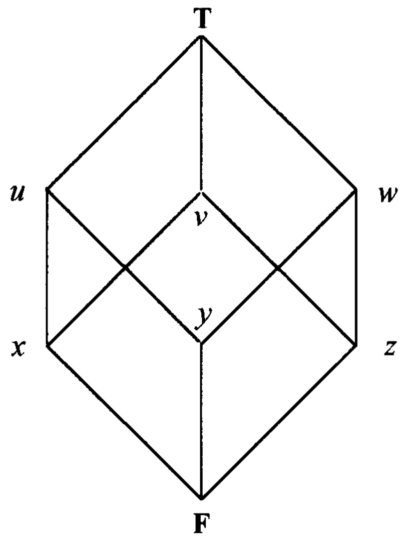
(1) $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ are connections from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$,
(2) not $a_1 Q b_1$,
(3) for all $c_1 \in B_1$ and $c_2 \in B_2$ it holds that if $c_1 R c_2$, then either
  (i) $c_1 R a_1$ and $a_2 R c_2$, or
  (ii) $c_1 R b_1$ and $b_2 R c_2$.

The notion of pair coupling is interesting, since the following holds. If $\{\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle\}$ is a pair coupling and $\langle c_1, c_2 \rangle$ is a connection, then either $c_1 Q a_1$ and $c_2 Q a_2$, or $c_1 Q b_1$ and $c_2 Q b_2$. Thus, "up to similarity", there are only the two connections $\langle a_1, a_2 \rangle$, $\langle b_1, b_2 \rangle$.

3.4. Connections and Ideals

In a *cis*, we can think of the specific normative content as expressed by connections. Connections can be said to reduce the number of elements of the domain of the quotient algebra determined by the Boolean quasi-ordering in view (and obtained by the transition to R-equivalence classes). In order to show how such a reduction works, a simple example might be useful. Let a particular *cis* be a Boolean quasi-ordering $\mathscr{B}$ with two fragments $\mathscr{B}_1$ and $\mathscr{B}_2$, where B_1 is disjunct from B_2. Thus for this *cis* let $B = \{a_1, a_2\}^*$, $B_1 = \{a_1\}^*$, $B_2 = \{a_2\}^*$. Furthermore, let the relation $\leqslant$ of the algebra $(B, \wedge, ')$ denote implications that hold due to logic alone, and let R_1 be the relation $\leqslant$ restricted to B_1 and R_2 the relation $\leqslant$ restricted to B_2. In other words, for the fragments $\mathscr{B}_1$ and $\mathscr{B}_2$ respectively, R_1 and R_2 denote implications that hold due to logic alone. Then there are sixteen elements in B, and each of B_1, B_2 has four elements. Now suppose that the pair $\langle a_1, a_2 \rangle$ belongs to R. (For example, this is due to a legal enactment.) We suppose that $R_1 \cup R_2 \cup \{\langle a_1, a_2 \rangle\} \subseteq R$, and that R contains as few pairs as possible. Next let us consider the quotient algebra of $\langle B, \wedge, ' \rangle$ with respect to Q. Then two observations can be made.

Firstly, the number of elements of the quotient algebra is eight. (This is due to the fact that one atom, namely $a_1 \wedge a_2'$ in $\langle B, \wedge, ' \rangle$, is "eliminated".) Graphically, the quotient algebra can be represented by the following cube:



$$\mathbf{F} = \{\bot, a_1 \wedge a_2'\},\ x = \{a_1, a_1 \wedge a_2\}, y = \{a_1' \wedge a_2, (a_1 \wedge a_2') \vee (a_1' \wedge a_2)\},$$

$$z = \{a_2', a_1' \wedge a_2'\},\ u = \{a_2, a_1 \vee a_2\},\ v = \{a_1 \vee a_2', (a_1 \wedge a_2) \vee (a_1' \wedge a_2')\},$$

$$w = \{a_1', a_1' \vee a_2'\},\ \mathbf{T} = \{\top, a_1' \vee a_2\}.$$

The second observation concerns connections and couplings from $\mathscr{B}_1$ to $\mathscr{B}_2$. The fact that, in this example, there is only one connection (hence a coupling) from $\mathscr{B}_1$ to $\mathscr{B}_2$, namely $\langle a_1, a_2 \rangle$, can be seen in the following way (cf. section 2.3). R corresponds to a proper congruence relation Q, and Q corresponds to an ideal I. Furthermore, since $a_1 R a_2$, it follows that $a_1 \wedge a_2' \in I$. Since R is assumed to contain as few pairs as possible, I shall be the least ideal that has $a_1 \wedge a_2'$ as an element. This least ideal is $\{\bot, a_1 \wedge a_2'\}$. The pair $\langle a_1, a_2 \rangle$ is a connection from $\mathscr{B}_1$ to $\mathscr{B}_2$ since $a_1 \wedge a_2'$ is a maximal element of I.[10] If there were more connections from $\mathscr{B}_1$ to $\mathscr{B}_2$ than $\langle a_1, a_2 \rangle$, there would be more maximal elements than $a_1 \wedge a_2'$ in I. Hence $\langle a_1, a_2 \rangle$ is the one and only connection from $\mathscr{B}_1$ to

[10] Suppose that, for all $x_1, y_1 \in B_1$ and all $x_2, y_2 \in B_2$, $x_1 \wedge x_2' < y_1 \wedge y_2'$ if and only if: $x_1 < y_1$ and $x_2' \leqslant y_2'$, or $x_1 \leqslant y_1$ and $x_2' < y_2'$. Then, given the assumption that $\mathscr{B}$ is finite and the fragments $\mathscr{B}_1$ and $\mathscr{B}_2$ are purely logical, the set of connections corresponds to the set of maximal elements of I with respect to $\leqslant$ that have the form $z_1 \wedge z_2'$ (where $z_1 \in B_1$ and $z_2 \in B_2$).

$\mathscr{B}_2$. Therefore, $\langle a_1, a_2 \rangle$ is a coupling from $\mathscr{B}_1$ to $\mathscr{B}_2$. (For analogous reasons, $\langle a'_2, a'_1 \rangle$ is a coupling from $\mathscr{B}_2$ to $\mathscr{B}_1$ in $\mathscr{B}$.)

3.5. Connections and Intermediaries

As appears from the preceding section, the notions of joining, connection, and coupling are of interest as regards some intended models of the theory. A case of special interest is the one where the relations R_1 and R_2 within fragments $\mathscr{B}_1$ and $\mathscr{B}_2$ are purely logical, and where the additional content is accomplished by connections. In this context, the notion of an "intermediary" is important.

Let us say that m is an *intermediary* from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$, if there exists a connection $\langle b_1, b_2 \rangle$ from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$ such that

$$b_1 RmRb_2$$

expresses the meaning of m.[11]

As an example, let us consider the condition "to be a citizen of the U. S." as an intermediary in a condition implication structure. Simplifying matters, we state the relevant conditions as follows:

b_1: to be a person born or naturalised in the U.S.
j_1: to be a person subject to the jurisdiction of the U.S.
s_2: to be sane,
a_2: to be adult,
e_2: to be entitled to vote.
c: to be a citizen of the U.S.

What holds according to legal rules is represented by:

(1) $(b_1 \wedge j_1)Rc$.
(2) $cR(s'_2 \vee a'_2 \vee e_2)$.

Let $\mathscr{B}$ be a Boolean quasi-ordering with two fragments $\mathscr{B}_1$ and $\mathscr{B}_2$, where $B_1 = \{b_1, j_1\}^*$ and $B_2 = \{s_2, a_2, e_2\}^*$, and where R_1 is $\leqslant$ restricted to B_1 and R_2 is $\leqslant$ restricted to B_2. Then, given the legal rules (1) and (2), $\langle b_1 \wedge j_1, s'_2 \vee a'_2 \vee e_2 \rangle$ is a connection from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$. Furthermore, if (1) and (2) give the meaning of c, then c is an intermediary from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$.

We now see that the two sentences (1) and (2) express parts of what determines that c is an intermediary between two fragments, one of which is primarily descriptive and the other contains so-called "hypothetical legal consequences". The use of intermediaries in this way is an important part

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[11] This does not exclude that $m \in B_1$ or $m \in B_2$. With a view to some applications, it may be appropriate to exclude this by further assumptions about $\mathscr{B}_1$ and $\mathscr{B}_2$.

of the legal technique of joining fragments of a legal system. In particular, it makes it possible to use intermediaries as so-called "vehicles of inference", or "inference tickets" in legal reasoning.[12]

Intermediaries are of different kinds. Depending on what *cis* is in view and how the fragments are chosen, an intermediary can be a component of a connection, or it can lie outside both fragments (as in the illustration of citizenship above). Furthermore, the connection that involves an intermediary can be a mere connection, or it can be a coupling, or it can be a pair coupling. A special case of an intermediary involved in a pair coupling is where the set $\{\langle a_1, a_2\rangle, \langle a_1', b_2\rangle\}$ is a pair coupling from $\mathscr{B}_1$ to $\mathscr{B}_2$ in $\mathscr{B}$, and $a_1 RmRa_2$, together with $a_1' Rm'Rb_2$, expresses the meaning of m. Thus, consider the condition c, "to be a citizen of the U.S.". For appropriately chosen fragments (and suitable conditions a_1, a_2, b_2), there can be a pair coupling $\{\langle a_1, a_2\rangle, \langle a_1', b_2\rangle\}$ such that $a_1 RcRa_2$ together with $a_1' Rc'Rb_2$ expresses the meaning of c.

4. Concluding Remarks

4.1. Classes of Boolean Quasi-orderings

In Lindahl and Odelstad 1999 we discussed the problem of "ground-open" intermediaries. We will finish this paper by some further remarks concerning this problem area.

In section 2.2 we suggested that a normative system might be represented by a *cis*, i.e., by a Boolean quasi-ordering with a domain of conditions. This, however, is a simplification, since it is more realistic to represent a normative system by a *class* of Boolean quasi-orderings (with domains of conditions). Such a representation provides better tools for expressing that a normative system usually contains norms that give room for different interpretations.

When a normative system is represented by a class of Boolean quasi-orderings, one member of the class is the core structure, representing the uncontroversial and settled implicative contents of the system. The other members of the class are conceived of as "amplifications" of the core. These amplifications represent different interpretations of the law, all of which conform with the core but which differ among themselves on issues which are not settled by the uncontroversial implicative contents. Thus the class representing a normative system consists of core and amplifications.

[12] On "vehicles of inference", and "inference tickets", cf. Wedberg 1951, p. 273, and Prior 1960, pp. 38 f. See also Ross 1951, p. 477, (English translation (1956–57), p. 820), and Lindahl and Odelstad 2000, pp. 272 ff.

In our approach, each member of the class that represents the normative system only represents what holds according to the "implication-affirming" statements of the system, i.e., statements of the kind "*a* implies *b*". What holds according to "implication-rejecting" statements (like "not: *a* implies *b*") is taken into account as a general restriction on the entire class of relational structures that represent the system.

4.2. Amplification and Change

When the idea of core and amplifications is applied to an actual normative system, such as part of the law of a country, it is of great importance to distinguish between amplification and change. The case where a Boolean quasi-ordering $\mathscr{B}^1$ represents an amplification of the core $\mathscr{B}$ of a normative system S should be kept apart from the case where $\mathscr{B}^1$ represents the core of a different, or "changed", normative system S_1.

Suppose that the following is an uncontroversial part of the legal system S: Obtaining a gift by deceitful assertions implies being liable to having the gift annulled. Let a_1 be the antecedent and let a_2 be the consequent in the example. Then a_1Ra_2 is true for the core and for all amplifications. Now suppose that the question arises: Does as well obtaining a gift through careless misrepresentation (condition b_1) imply being liable to having the gift annulled? Suppose that this question is unsettled. This means that neither is b_1Ra_2 true for all amplifications of the system, nor is there a restriction not: b_1Ra_2 on the system.

Two different developments of the example are to be distinguished. Firstly, we have the course of events where the law is changed by an act of legislation or by judicial precedent: the normative authority stipulates that b_1 implies a_2 or it stipulates that not: b_1 implies a_2. Then the original system S is replaced by a system S_1 or by a system S_2, respectively. If the enactment is to the effect that b_1 implies a_2, then we get a system S_1 such that $b_1R^ia_2$ is true for all amplifications i of the system S_1. If the enactment is to the effect that not b_1 implies a_2, then we get a system S_2 where "not: $b_1R^ja_2$" is a restriction on all amplifications j of S_2.

The course of events involving a "change" in the sense now described should be distinguished from another course of events where the system S remains unaltered. Suppose that a subordinate court or authority, not having the power to change the law, pronounces that b_1 implies a_2 or that not: b_1 implies a_2. Then the law is not changed. The statement $b_1R^ia_2$ is still not true for all amplifications i of S, and there is still no restriction on all these models to the effect that not: $b_1R^ia_2$.

It should be added that, in the example, the subordinate court or authority has not committed any judicial error if it chooses to act on an amplification, say $\mathscr{B}^k$, such that $b_1R^ka_2$ is true for that amplification. But neither has the subordinate court or authority committed any judicial error if it rejects the amplification $\mathscr{B}^k$. In either case the judge or official is not liable to being prosecuted for breach of duty.

The assumption that, in the way just described, there can be opposite courses of action neither of which involves judicial error, does not imply that all of these courses of action are indifferent. With regard to a system S, there can be good reasons for claiming that one amplification of S is better than another one, or that a particular restriction ought to be put on all amplifications of S. The case that there are plausible claims of this nature, however, should be distinguished from the case that S is in fact replaced by a different system S_1 or S_2.[13]

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[13] Some legal philosophers (cf. Dworkin 1977) would claim that in well-developed legal systems there is always just one right answer to questions of law. This position can be expressed as the combination of two claims. One is the claim that among what we call amplifications there is always a best one. The other is the claim that the preference ordering among amplifications (or the set of principles which, when applied, yield this preference ordering) is itself a part of what should be called "the law". In this paper, we do not deal with the questions connected with these two claims.

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